

BEAMS

Thin-walled beams of open cross section

Beams, whose thickness of individual partial cross section part t_i is small comparing to beam cross section dimensions (b, h) of we call thin-walled. Literature specifies rate app. 1/10.

There are differentiated open and closed cross sections, according to fact whether centre line constitutes closed curve. Thereinafter we will deal only with open cross sections according to theory of Prof. Vlasov. There belong e.g. I, C, T, U sections.

Theory is based on 2 assumptions:

1/ Deformation of cross section outline in its plane does not exist. It follows that cross sections twist in their plane as rigid aggregates.

2/ Elements of centre line surface originally rectangular, remain after deformation rectangular, as well, i.e. their angular deformation is nought.

If the element is stressed only by torque, in the element could originate:

a/ only tangential stresses. This phenomenon originates at free torsion (Saint-Venant), diamonding (deplanation) of cross section is not restrained, it could freely wage.

b/ tangential stresses and normal stresses. Most frequent case – free torsion is restrained by boundary condition, loading, change of torque value along length of the beam, or change of cross section along length of the beam. We call this type as bounded torsion. We neglect it at massive cross sections, at thin-walled it play significant role on overall elasticity of the beam.

Bounded torsion could arise also due to other types of loading, it need not be just a torque. It could be also shear force that does not pass through centre of shear, or axial force acting apart from centre of gravity of the cross section.

Vector of deformation: $u = [u, in, w, \omega_x, \omega_y, \omega_z, \theta]^T$

Differential equations of equivalence:

$$M_{sv} = G.I_T \cdot \frac{d\omega_x}{dx}$$

$$B = -E.I_w \cdot \frac{d^2\omega_x}{dx^2}, \text{ where}$$

$$M_w = -E.I_w \cdot \frac{d^3\omega_x}{dx^3}$$

M_{sv} is a free torsion (St.Venant),

M_w is a bounded torsion (Vlasov, warping),

B is bending torsion bi-moment.

Total torque is: $M_x = M_{sv} + M_w$. After substitution from antecedent equations we receive differential equations of torsion:

$$M_x = G.I_T \cdot \omega' - E.I_w \cdot \omega'''$$

Number of nodes:

2

Shape functions:

$$u = a_1 + a_2.x$$

$$in = b_1 + b_2.x + b_3.x^2 + b_4.x^3$$

$$w = c_1 + c_2.x + c_3.x^2 + c_4.x^3$$

$\varphi = d_1 + d_2.x + d_3.x^2 + d_4.x^3$, for thick cross-section is used only linear term:

Stability of Thin-walled beams

Example c.xx

Critical force of unequal angle, articulately fixed. Length $L=2$ [m], $E=210$ [GPa], $G=84$ [GPa]. Cross section rectangular, free of slopes and radiuses, thickness of arms 10[mm], length centreline arms 250[mm] and 150[mm].

Numeric solution and solution according to Vlasov approaches with increasing length of beam the Euler critical force.

	analytical Euler	analytical Vlasov	1D model	2D model
P_{crit} [kN]	2250	650	632	658

Table No.1
Results.

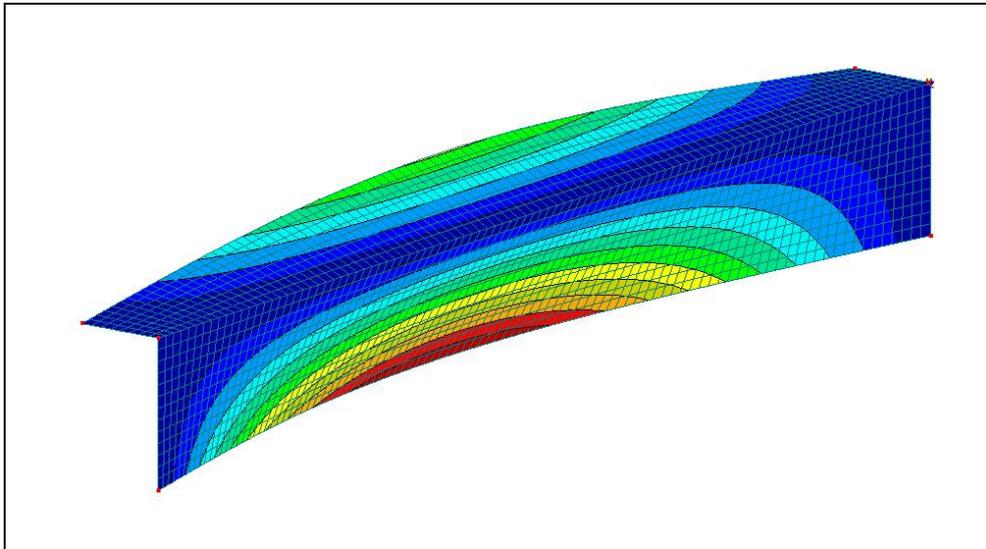


Fig. c. xx

Shape of buckling with torsion. Modelled by 2D shell elements.

PLANAR ELEMENTS

Coordinate systems

Global coordinate system (abbreviation: GCS)

It is a positive dextrotatory coordinate system valid for entire construction. In GCS there are computed deformations of nodes.

It is denoted as $[x^G, y^G, z^G]$.

Plenary coordinate system (abbreviation: PCS)

It is a coordinate system of planar element, each element has its PCS. It is a positive rectangular, dextrotatory system. Axis z^P is perpendicular to plane element and planar axes x^P, y^P lay in the plane element. Definition of positive axis x^P : it is parallel with intersection of the plane x^G, y^G with the plane of element, and its positive sense is determined in such a way that axes $+x^G, +x^P$ contain together acute angle $\langle 0^\circ, 90^\circ \rangle$. This definition becomes uncertain, if:

1/ does not exist intersection of planes - element lay parallel with x^G, y^G . Axis $+x^P$ is determined according to rule on acute angle between $+x^G$ and $+x^P$.

2/ intersection exist, but axis x^G is perpendicular on plane of element - element lay in plane parallel with y^G, z^G .

In both exceptional cases is applied rule on determination of $+x^P$ by means of acute angle between $+x^G$ and $+x^P$.

In the PCS there are computed internal forces.

They are denoted as $[x^P, y^P, z^P]$.

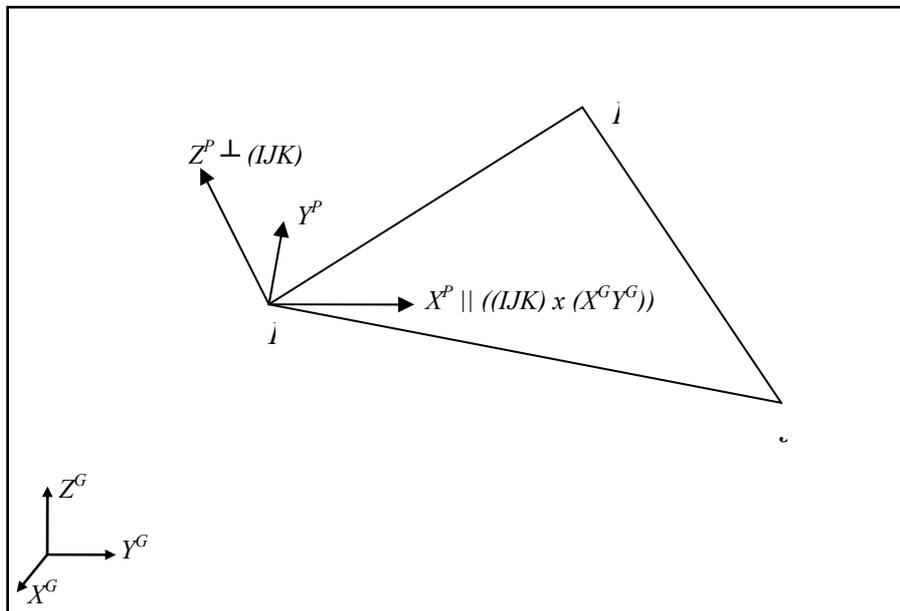


Fig. c. xx

Plenary co-ordinate system of planar element.

Mark convention of internal forces

Each internal force is an integral thrust line of corresponding component of stresses.

Membrane (wall) internal forces.

$$N_x = \int_h \sigma_x \cdot dz$$

$$N_y = \int_h \sigma_y \cdot dz$$

$$Q_{xy} = \int_h \tau_{xy} \cdot dz$$

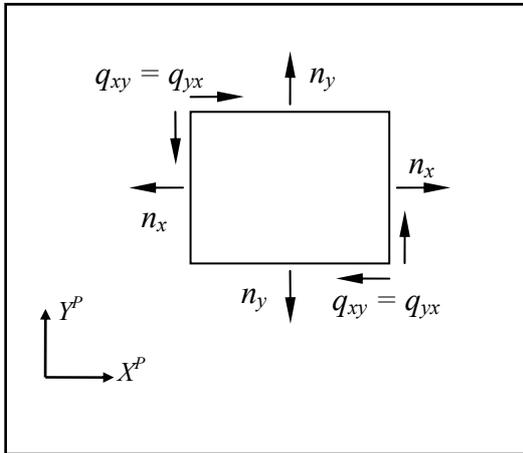


Fig. c. xx
Positive membrane force element.

Bend (plate) internal forces.

$$M_x = \int_h \sigma_x \cdot z \cdot dz$$

$$M_y = \int_h \sigma_y \cdot z \cdot dz$$

$$M_{xy} = \int_h \tau_{xy} \cdot z \cdot dz$$

$$Q_x = \int_h \tau_{zx} \cdot dz$$

$$Q_y = \int_h \tau_{yz} \cdot dz$$

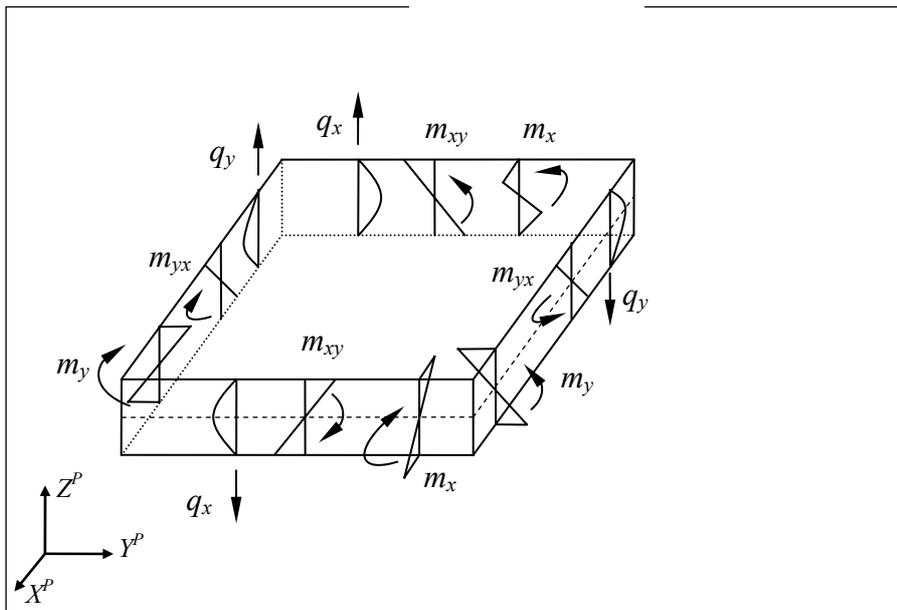


Fig. c. xx
Positive bend force element.

Transformations

Transformation matrix for 2D elements in space:

$$T = \begin{bmatrix} L_1^2 & M_1^2 & N_1^2 & 2.L_1.M_1 & 2.M_1.N_1 & 2.N_1.L_1 \\ L_2^2 & M_2^2 & N_2^2 & 2.L_2.M_2 & 2.M_2.N_2 & 2.N_2.L_2 \\ L_3^2 & M_3^2 & N_3^2 & 2.L_3.M_3 & 2.M_3.N_3 & 2.N_3.L_3 \\ L_1.L_2 & M_1.M_2 & N_1.N_2 & L_1.M_2 + L_2.M_1 & M_1.N_2 + M_2.N_1 & N_1.L_2 + N_2.L_1 \\ L_2.L_3 & M_2.M_3 & N_2.N_3 & L_2.M_3 + L_3.M_2 & M_2.N_3 + M_3.N_2 & N_2.L_3 + N_3.L_2 \\ L_3.L_1 & M_3.M_1 & N_3.N_1 & L_3.M_1 + L_1.M_3 & M_3.N_1 + M_1.N_3 & N_3.L_1 + N_1.L_3 \end{bmatrix}, \text{ where}$$

$$\begin{aligned} L_1 &= \cos(x^G, x^P) & M_1 &= \cos(y^G, x^P) & N_1 &= \cos(z^G, x^P) \\ L_2 &= \cos(x^G, y^P) & M_2 &= \cos(y^G, y^P) & N_2 &= \cos(z^G, y^P) \\ L_3 &= \cos(x^G, z^P) & M_3 &= \cos(y^G, z^P) & N_3 &= \cos(z^G, z^P) \end{aligned}$$

Transformation matrix for 2D task in plane:

Wall element – arise of previous matrix $T_{[6.6]}$, dropping out of corresponding rows and columns that belong to components that are not defined in this task. They are components: $[\sigma_z, \tau_{yz}, \tau_{zx}]$.

$$T_M = T_B = \begin{bmatrix} L_1^2 & M_1^2 & 2.L_1.M_1 \\ L_2^2 & M_2^2 & 2.L_2.M_2 \\ L_1.L_2 & M_1.M_2 & L_1.M_2 + L_2.M_1 \end{bmatrix}$$

Plate elements – again there is taken, as base, full transformation matrix $T_{[6.6]}$, in what is not defined only σ_z . For bending and torsion part $T_{B[3.3]}$ could be used matrix for plane elasticity $T_{M[3.3]}$. If the plate is with lateral shear, for shear force will be applied, regarding to keeping of direction $N_3 = 0$, transformation matrix $T_{S[2.2]}$:

$$T_S = \begin{bmatrix} M_2 & L_2 \\ M_1 & L_1 \end{bmatrix} \text{ that arise by modification of 5}^{\text{th}} \text{ and 6}^{\text{th}} \text{ row and column of original}$$

transformation matrix $T_{[6.6]}$. Regarding to sequencing of components of deformation vector: $\varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T$, the sequence of vector components of stresses: $\sigma = [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}]^T$. To the stress τ_{yz} belongs Q_y , to the stress τ_{zx} belongs Q_x . Than are shear force computed in sequence: $[Q_y, Q_x]$.

Transformation of physical qualities

$$\begin{aligned} D^p &= (T^I)^T \cdot D \cdot T^I \\ D &= T^I \cdot D^p \cdot T \end{aligned}$$

Transformation of relative deformation

It is possible to transform through the same transformation matrix as the vector of stress, only tensor of deformation. Vector of deformation is possible, but it is necessary to modify its components. For shear components of deformation vector should be substituted half value of full technical slope that ε have nature of tensor, with what is possible further work, as is usual

at vectors. Parameter γ means change of original right angle among directions of axis specified in indexes.

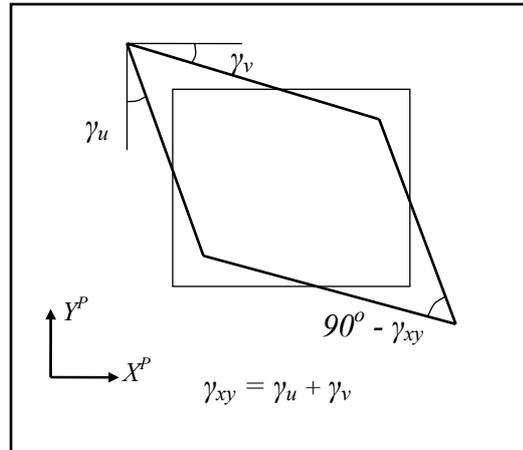


Fig. c. xx

Relatively angular deformation – of slope.

$$\begin{aligned}\varepsilon^p &= T \cdot \varepsilon \\ \varepsilon &= T^{-1} \cdot \varepsilon^p\end{aligned}$$

Transformation of the stress tensor

$$\begin{aligned}\sigma^p &= T \cdot \sigma \\ \sigma &= T^{-1} \cdot \sigma^p\end{aligned}$$

Planar elasticity and deformation

Geometry is defined in plane. It is possible to solve both cases of planar elasticity:

- planar elasticity – loading, as well as reactions act in wall plane. Vector of stress: $\sigma = [\sigma_x, \sigma_y, 0, \tau_{xy}, 0, 0]^T$, after substitution to physical equations:

$$\begin{aligned}\varepsilon_x &= \frac{1}{E} \cdot [\sigma_x - \mu \cdot (\sigma_y + \sigma_z)] & \gamma_{xy} &= \frac{1}{G} \cdot \tau_{xy} \\ \varepsilon_y &= \frac{1}{E} \cdot [\sigma_y - \mu \cdot (\sigma_x + \sigma_z)] & \gamma_{yz} &= \frac{1}{G} \cdot \tau_{yz} \\ \varepsilon_z &= \frac{1}{E} \cdot [\sigma_z - \mu \cdot (\sigma_x + \sigma_y)] & \gamma_{zx} &= \frac{1}{G} \cdot \tau_{zx}\end{aligned}$$

we will reach vector of deformation: $\varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, 0, 0]^T$. Deformation $w_{(x,y,z)}$ in direction upright to the plane of plate exist, but it is irrelevant. Wall in this direction freely change thickness, nothing prevent this.

- planar deformation - layer with thickness $h = l$ cut from the points that does not allow deformations in direction upright to the wall plane. Vector of deformation: $\varepsilon = [\varepsilon_x, \varepsilon_y, 0, \gamma_{xy}, 0, 0]^T$. We will reach stress vector through inversion of physical equations: $\sigma = [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, 0, 0]^T$. Wall tray to deform also upright to its plane, what is prevented by adjacent layers.

In next we work with reduced vectors of deformation and stresses:

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} = \tau_{yx} \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Missing components is possible to computed of physical equations. Thickness h within one element is constant.

In solution of geometrically non-linear tasks by the Neton-Raphson method is used quadratically only Green vector of deformation ε_{II} that comprises non-linear terms that are in theory of small deformation neglected:

$$\varepsilon_{II} = \begin{bmatrix} \frac{1}{2} \cdot \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \\ \frac{1}{2} \cdot \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\ \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y} \end{bmatrix}$$

Membrane isoparametric elements with rotational degrees of freedom (Serendipity family)

Each vertex of element has 3 degrees of freedom 2 translations in wall plane walls and rotation: $u = [u, in, \omega_z]$. Vector of deformation and corresponding geometric equations that comprise only linear part:

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

Element could be physically isotropic, or orthotropic. Matrix of physical constant D_M :

$$D_M = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ sym. & & D_{33} \end{bmatrix}$$

In case of isotropy is valid:

$$D_{13} = D_{23} = 0$$

planar elasticity:

planar deformation:

$$D_{11} = D_{22} = \frac{E}{1 - \mu^2}$$

$$D_{11} = D_{22} = \frac{E \cdot (1 - \mu)}{(1 + \mu) \cdot (1 - 2 \cdot \mu)}$$

$$D_{33} = \frac{E}{2 \cdot (1 + \mu)}$$

$$D_{33} = \frac{E \cdot (1 - \mu)}{2 \cdot (1 + \mu) \cdot (1 - 2 \cdot \mu)}$$

$$D_{12} = \frac{E \cdot \mu}{1 - \mu^2}$$

$$D_{12} = \frac{\mu \cdot E \cdot (1 - \mu)}{(1 + \mu) \cdot (1 - 2 \cdot \mu)}$$

Material orthotropy is given by angle of rotation - angle β of orthotropy to planar co-ordinate element system.

Vector of deformation in case thermal loading: $\varepsilon = [\alpha \cdot T, \alpha \cdot T, 0]^T$ where T is change of temperature from stadium of preparation (zero elasticity) after actual status 1.

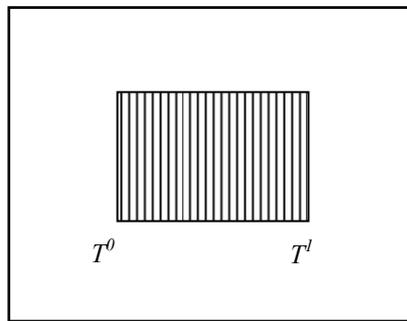


Fig. c. xx

Uniform change of temperature.

Result of solution are at:

- planar elasticity of internal forces n (normal and shear):, $n = \begin{bmatrix} n_x \\ n_y \\ q_{xy} = q_{yx} \end{bmatrix}$

- planar deformation of stresses σ (normal and shear):. $\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} = \tau_{yx} \end{bmatrix}$

Numeric integration of matrices

There is utilised Gauss numeric integration that converts integral to sum of products of weighted coefficients and functional values of integrated function. Number of integration points and their position is for each element different; their exact description is presented at description of each element. At composition of matrix of elements by numeric integration is not allowed to change its parameters, numeric integration is firmly bounded with each type of element and there is selected such number of integration points and their position that provides best results comparing to amount of machine instructions. Integration on unit square is:

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^m w_i \cdot w_j \cdot f(\xi_i, \eta_j), \text{ where}$$

m is a number of integration points,
 w are weight coefficients,
 $f(\xi, \eta)$ are functional of value of shape function in corresponding point.

For 3-node element is used analogical procedure, boundary of integral are $\langle 0, 1 \rangle$.

Elimination of nodes in centre of sides

Field of deformation parameters of 3-nodal and 4-nodal element is approximated by quadratic multinomial, original geometry is with node in centre of each side. On each side of the element is anticipated that is straight. This enables to eliminate central node, what is moreover favourable from aspect of width of the system of equations and thus consumption of machine time. Resulting geometry of the element is described freely of centre node – it is linear. Resulting element is than sub-parametrical.

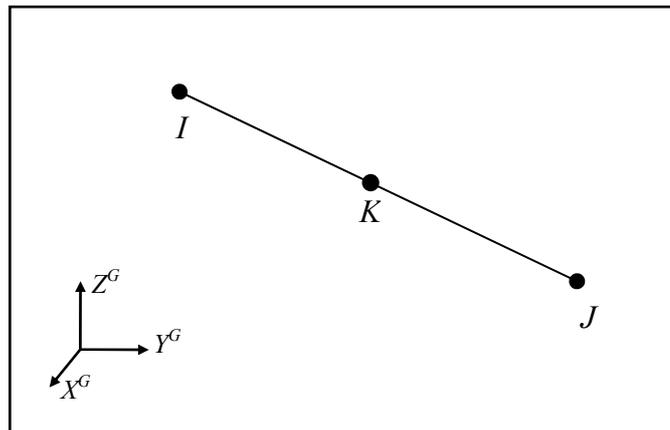


Fig. c. xx
 Edge of element with centre node.

Original elements have nodes in in centres of sides $[u_k, v_k]$, these nodes are condensed by transformation of deformation parameters to vertices by interpolation:

$$u_k = \frac{1}{2} \cdot (u_i + u_j) + \frac{y_j - y_i}{8} \cdot (\omega_{z_j} - \omega_{z_i})$$

$$v_k = \frac{1}{2} \cdot (u_i + u_j) + \frac{y_j - y_i}{8} \cdot (\omega_{z_j} - \omega_{z_i})$$

There will arise element that is compatible with other finite elements 1D and 3D that have corresponding degrees freedom in rotation.

Spurious mode control

For isoparametric elements is necessary to numerically stabilize elements, to avoid zero-energy modes. In case of 3-nodal element there appears only equal rotation of spurious mode, in case of 4-nodal element there is also hourglass mode. Stabilisation consists in adding of small

contribution to the potential energy to E element. Size of the parameter α_R and α_H is not possible to be changed.

1/ Hourglass mode

It occurs only at 4-nodal beams if there is used reduced integration. This mode vanishes, if there is used disordered net, or more elements.

There is valid: $\omega_{z1} = -\omega_{z2} = \omega_{z3} = -\omega_{z4}$

Element has added energy E_H :

$$E_H = \alpha_H \cdot IN \cdot \omega_H \cdot G_{xy} \cdot \omega_H$$

α_H - parameter of energy ($1 \cdot 10^{-3}$)

V - element volume

$$\omega_H = 1/4 \cdot (\omega_{z1} - \omega_{z2} + \omega_{z3} - \omega_{z4})$$

G_{xy} - modulus of elasticity in shear

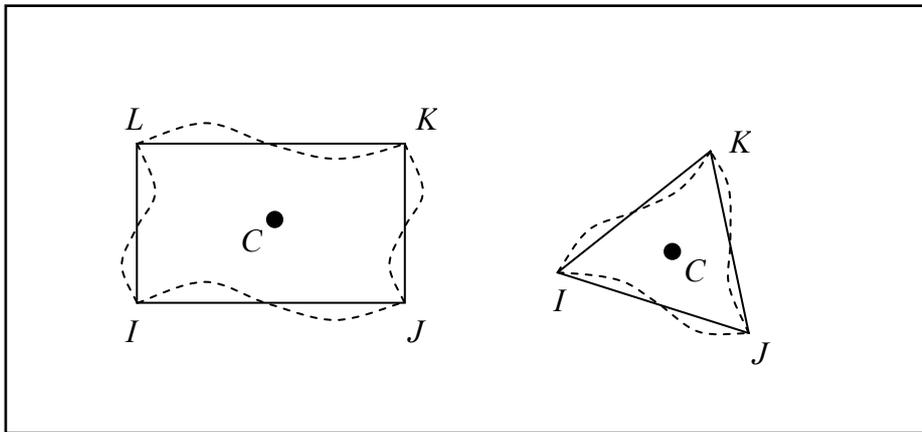


Fig. c. xx

Equal rotation spurious mode.

2/ Equal rotation mode

It occurs at 3-nodal and 4-nodal elements.

There is valid: $\omega_{z1} = \omega_{z2} = \omega_{z3} = \omega_{z4}$

Element have added energy E_R :

$$E_R = \alpha_R \cdot IN \cdot \omega_R \cdot G_{xy} \cdot \omega_R$$

α_R - parameter of energy ($1 \cdot 10^{-6}$)

V - element volume

ω_R - relative rotation, computed in the centre of the C element:

$$\omega_R = \omega_C - \frac{1}{n} \cdot \sum_{i=1}^n \omega_{zi}, \text{ where } \omega_C = \frac{1}{2} \cdot \left(\frac{\partial v}{\partial x_C} - \frac{\partial u}{\partial y_C} \right)$$

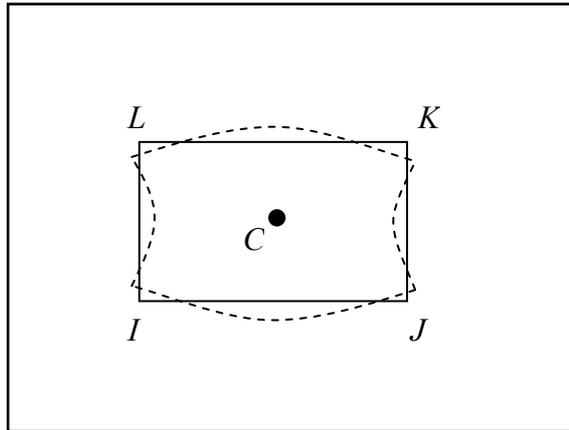


Fig. c. xx
Hourglass spurious mode.

Air elements

Special type of element that is possible to be used in case of validity of state equation of gases. They are constructions that are closed, filling if constituted by gas that at the same time assures synergism. It could to be insulation glass, where among several solid layers of glass and case of glued foil is present of given production temperature and pressure. We assume that temperature of element is constant, there is applied isothermal gas equation:

$$p_1 \cdot V_1 = p_2 \cdot V_2$$

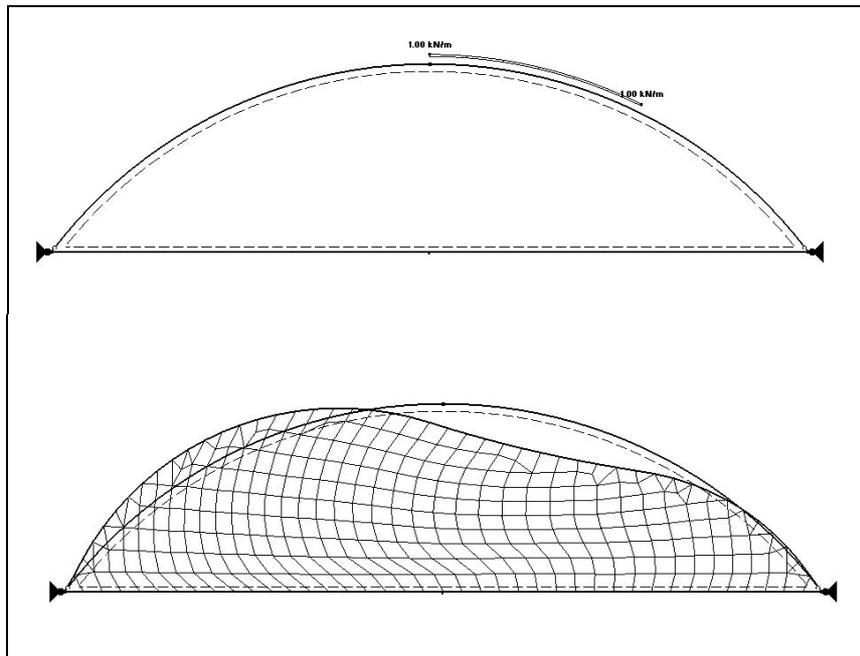


Fig. c. xx
Air hall solidly supported along bottom edge, asymmetrically loaded by uniform force loading, geometry, deformed shape.

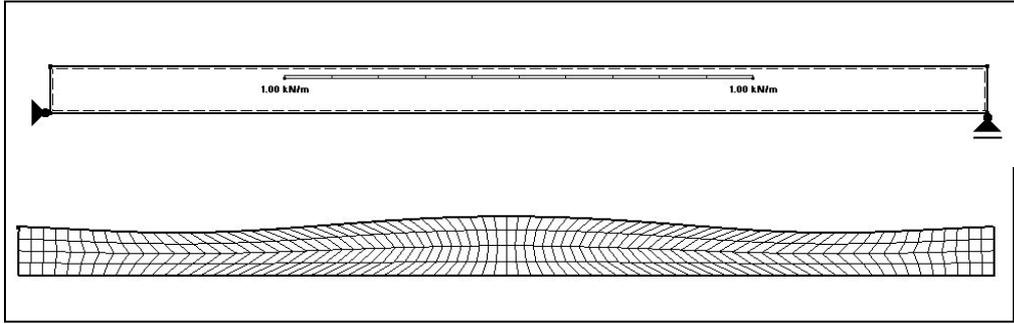


Fig. c. xx

Fragment of insulation construction with bottom solid edge and upper soft edge, symmetrically loaded with uniform force loading (underpressure), geometry and deformed shape.

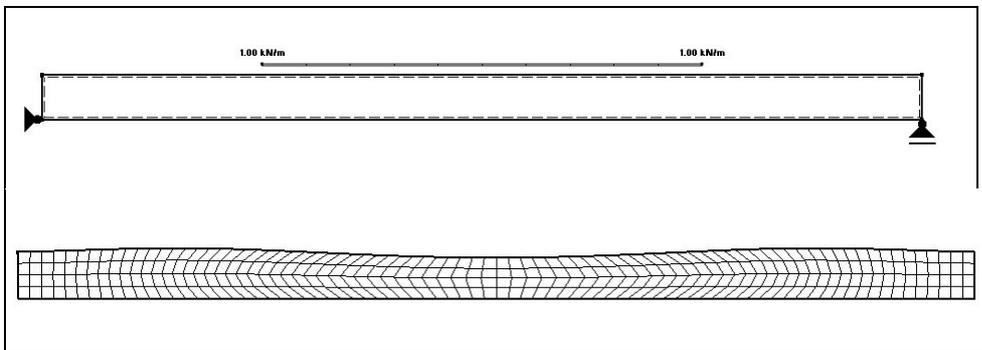


Fig. c. xx

Construction of previous example, loaded by symmetrically uniform force load (overpressure), geometry and deformed shape.

4-nodal element:

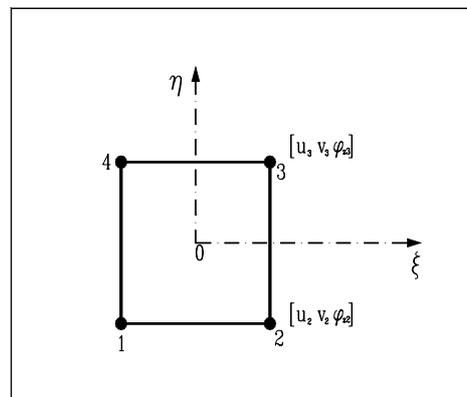
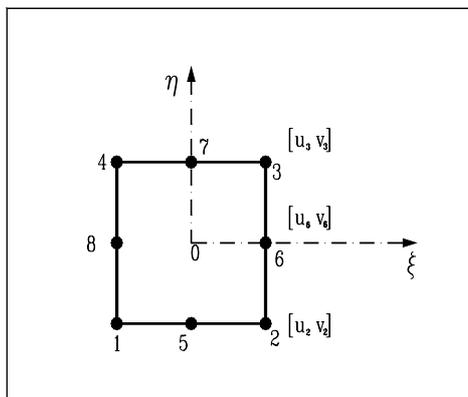


Figure 1, 2

8-nodal origin 16 DOF membrane without RDOF.

4-nodal new 12 DOF membrane with RDOF.

Number of nodes:

Unit coordinates of nodes:
 $\{-1, -1\}, \{1, -1\}, \{1, 1\}, \{-1, 1\}$
Shape functions:

$$N_1 = \frac{1}{4} \cdot (1 - \xi) \cdot (1 - \eta) \cdot (-\xi - \eta - 1)$$

$$N_2 = \frac{1}{4} \cdot (1 + \xi) \cdot (1 - \eta) \cdot (\xi - \eta - 1)$$

$$N_3 = \frac{1}{4} \cdot (1 + \xi) \cdot (1 + \eta) \cdot (\xi + \eta - 1)$$

$$N_4 = \frac{1}{4} \cdot (1 - \xi) \cdot (1 + \eta) \cdot (-\xi + \eta - 1)$$

$$N_5 = \frac{1}{2} \cdot (1 - \xi^2) \cdot (1 - \eta)$$

$$N_6 = \frac{1}{2} \cdot (1 + \xi) \cdot (1 - \eta^2)$$

$$N_7 = \frac{1}{2} \cdot (1 - \xi^2) \cdot (1 + \eta)$$

$$N_8 = \frac{1}{2} \cdot (1 - \xi) \cdot (1 - \eta^2)$$

Numeric integration:

Reduced (2 x 2)

integration points and weight multiplier:

$$\xi = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

$$\eta = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$w = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Evaluation:

Function of surface on extrapolation of qualities of integration points to peaks:

$$F = a_1 + a_2 \cdot \xi + a_3 \cdot \eta + a_4 \cdot \xi \cdot \eta$$

Loading:

Element could be loaded:

- of node force (moment) in the node,
- uniform loading on edge,
- volume loading,
- relative deformation of surface ε caused by uniform warming up, eventual uniform shrinkage.

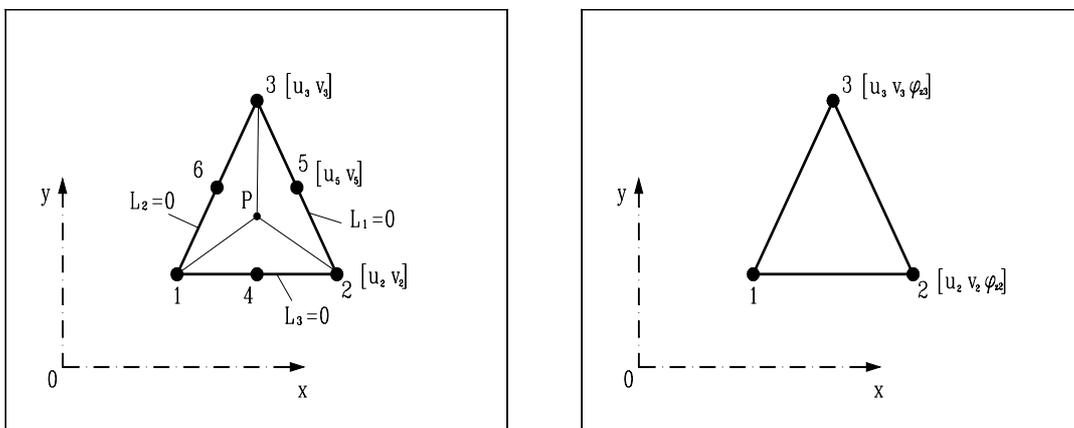
3-nodal element:

Figure 3, 4

6-nodal origin 12 DOF membrane without RDOF.

3-nodal new 9 DOF membrane with RDOF.

Unit coordinates of nodes:

$\{0, 0\}, \{1, 0\}, \{0, 1\}$

Shape functions:

$$N_1 = 1 - 3 \cdot \xi - 3 \cdot \eta + 2 \cdot \xi^2 + 4 \cdot \xi \cdot \eta + 2 \cdot \eta^2$$

$$N_2 = -\xi + 2 \cdot \xi^2$$

$$N_3 = -\eta + 2 \cdot \eta^2$$

$$N_4 = 4 \cdot (\xi - \xi^2 - \xi \cdot \eta)$$

$$N_5 = 4 \cdot \xi \cdot \eta$$

$$N_6 = 4 \cdot (\eta - \xi \cdot \eta^2 \cdot \eta)$$

Numeric integration:

Selective reduced (4-point), 3 for $\varepsilon_x, \varepsilon_y$, 1 for γ_{xy}

integration points and weight multiplier:

$$\xi = \begin{bmatrix} 1/6 & 2/3 & 1/6 & 1/3 \end{bmatrix}$$

$$\eta = \begin{bmatrix} 1/3 & 1/3 & 2/3 & 1/3 \end{bmatrix}$$

$$w = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1 \end{bmatrix}$$

Evaluation:

$$F = a_1 + a_2 \cdot \xi + a_3 \cdot \eta$$

Normal stresses are extrapolated of integration points to vertices according to the function F . Shear stress is constant, there is evaluation centre of gravity of the element.

Matrices of finite elementsStiffness matrix

Dependency among vector field of deformation ε and unknown nodes by deformation parameters u is expressed by: $\varepsilon = \delta \cdot N \cdot u = B_L \cdot u$. Linear deformation matrix B is obtained by derivation of shape functions N .

Matrix of differential operators:

$$\delta^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

Matrix of shape functions:

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots \\ 0 & N_1 & 0 & N_2 & \dots \end{bmatrix}$$

Of potential energy of internal forces Π_i : $\Pi_i = \frac{1}{2} \int_{\Omega} \varepsilon^T \cdot \sigma \cdot d\Omega$ is obtained stiffness matrix of

$$\text{element } K_L: |K_L - \lambda \cdot K_G|_{\det} = 0$$

Geometric matrix, Matrix of influence of initial stress on element stiffness

This matrix is used for solution of problem of stability:

or for compilation of total tangency matrix of element with influence of elasticity:

$$K_T = K_L + K_G .$$

Non-linear part B_{NL} , taking into account influence of stress on overall stiffness of the element:

$$B_{NL} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots \\ \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \dots \end{bmatrix}$$

Cauchy stress matrix of previous iteration, in case of geometric non-linearity, or initial stresses – pre-stresses:

$$\tilde{\tau} = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix}$$

Then, geometric matrix of the element K_G is: $K_G = \int_{\Omega} B_{NL}^T \cdot \tilde{\tau} \cdot B_{NL} \cdot d\Omega$

Plate elements

Planar constructions, whose all points lay in one common plane. Thickness h is small, comparing to dimensions of the plate. At determination of characteristic dimension L of the plate (what is in case of rectangular plate a shorter dimension, in case of round plate diameter) it is possible to set up solutions of plate according to corresponding theory to:

1/ Thin plate – ratio $1/50 < h/L < 1/10$ - technical theory of bend of thin plates, based on Kirchhoff's assumptions. There belong almost all ceiling plates.

2/ Thick plate - ratio $1/10 < h/L < 1/5$ – Reissner-Mindlin's theory that respects shear of normal line slope due to shear deformation γ_{xz} , γ_{yz} . There could be used also thin plates, while it is necessary to eliminate shear of solving of cross section at small thicknesses.

While the ratio is $h/L > 1/5$ - construction should be already evaluate as 3D task, at ratio $h/L < 1/50$ the construction is necessarily to be solved a membrane, what is possible in the shell model, or membrane force are accentuated and in overall energetic potential could not be neglected.

Force loading acts upright to the plane of the plate, moment around axes of the plane plate. Normal line to the centreline plane of the element remains after deformation straight, it depends on theory of calculation, whether it remain perpendicular to the centreline plane (Kirchhoff's theory), or not (Reissner-Mindlin's theory).

Each vertex of element has 3 degrees of freedom: deflexions and rotation: $u = [w, \omega_x, \omega_y]$.

Vector of deformation in case of thermal loading: $\varepsilon = [\alpha \cdot \Delta T/h, \alpha \cdot \Delta T/h, 0, 0, 0]^T$ where ΔT is difference of temperature on upper and bottom surface.

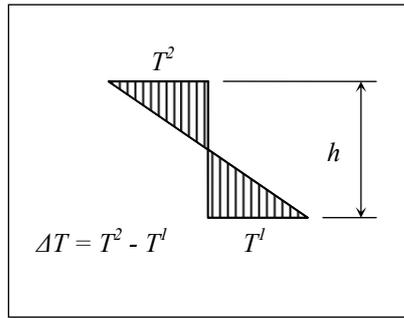


Fig. c. xx

Non-uniform change of temperature.

Result of solutions are internal moments and force:

$$m = \begin{bmatrix} m_x \\ m_y \\ m_{xy} = m_{yx} \\ q_x \\ q_y \end{bmatrix}$$

Thickness of element is constant.

Loading:

Element could be loaded:

- by node force (moment) in vertex,
- by uniform force loading on edge,
- by volume loading,
- by curvature of surface κ caused by non-uniform warming up, eventually non-uniform shrinkage.

Application of affect of flexible subsoil.

Flexible subsoil with shearing spreading of the Kolar-Nemec type could be easily added to the stiffness matrices. Stiffness matrix of subsoil is added to the stiffness matrix of the plate element. Potential energy of subsoil is given by:

$$\Pi_p = \frac{1}{2} \int_{\Omega} (C_1 \cdot w^2 + C_{2x} \cdot w'^2 + C_{2y} \cdot w''^2) d\Omega$$

Thin plates

Element is based on Kirchhoff theory of thin plate, where is neglected shear deformation - slope. Deformations of plate are described by 1 function of 2 variables $w_{(x,y)}$. Rotations are derivations of the deflexions. Out of assumptions of this theory there is valid $\varepsilon = [\varepsilon_x, \varepsilon_y, \theta, \gamma_{xy}, 0, 0]^T$.

Normal line to non-deformed surface of the plate remains a normal line also after its deformation, and under assumption of zero displacement u , in points of centreline surface there is valid:

$$u = z \cdot \varphi_x = z \cdot \frac{\partial w}{\partial x}$$

$$v = -z \cdot \varphi_y = -z \cdot \frac{\partial w}{\partial y}$$

After substitution to the geometric equations:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

we will reach vector of deformation ε that varies linearly by high of cross section h . We describe this vector for short by the vector of curvature of deflexions surface κ , what is for plate more convenient inscription:

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} = z \cdot \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial v}{\partial y} = -z \cdot \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2 \cdot z \cdot \frac{\partial^2 w}{\partial x \cdot \partial y} \end{bmatrix} \quad \kappa = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2 \cdot \frac{\partial^2 w}{\partial x \cdot \partial y} \end{bmatrix}$$

Between both vectors there is valid relation: $\varepsilon = z \cdot \kappa$

Normal stress σ_z is comparing to stresses σ_x and σ_y negligible small, they are not considered. Vector of stress is $\sigma = [\sigma_x, \sigma_y, 0, \tau_{xy}, 0, 0]^T$. Out of condition of zero slope there are zero also shear of component τ_{yz} and τ_{zx} . Discrepancy of assumptions with reality of this theory is in:

$$1/ \quad \varepsilon_z = \frac{1}{E} \cdot [-\mu(\sigma_x + \sigma_y)] \text{ physical equations, but } \sigma_z = 0.$$

$$2/ \quad \gamma_{yz} = \frac{1}{G} \cdot \tau_{yz} = 0, \text{ but out of conditions of equilibrium there is valid that course of}$$

$$\gamma_{zx} = \frac{1}{G} \cdot \tau_{zx} = 0$$

shearing stresses by high of cross section is parabolic.

Number of nodes:

3

Coordinates of nodes:

$\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}$ in planar co-ordinates

Deflexions function of the plate:

$$w = a_1.L_1 + a_2.L_2 + a_3.L_3 + a_4.L_1^2.L_2 + a_5.L_2^2.L_3 + a_6.L_3^2.L_1 + a_7.L_1^2.L_3 + a_8.L_2^2.L_1 + a_9.L_3^2.L_2 + 2.a_{10}.L_1.L_2.L_3$$

Meaning of parameters a_{1-9} : deflexions w and rotations ω_x, ω_y in each node. Parameter a_{10} is linear combination of parameters: $a_{10} = (a_4 + a_5 + a_6 + a_7 + a_8 + a_9) / 4$. To this parameter appertain deflexions w in the centre of gravity of element, we work in calculation free of internal nodes, and that is why we eliminate this parameter.

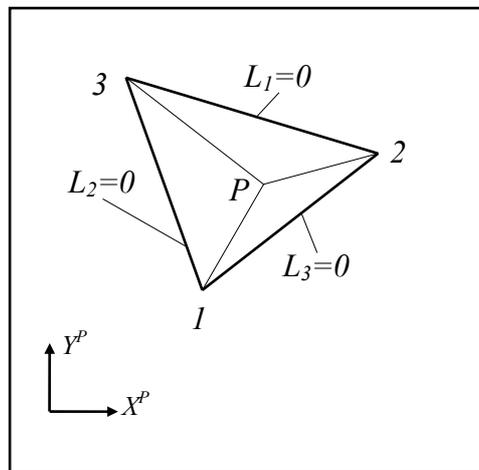
L_1, L_2, L_3 are planar coordinates. At 3-nodal element it is more convenient to work in the system of planar co-ordinates, as in the Cartesian system, due to calculation of integrals. Each planar co-ordinate L_i of the item P expresses ration of surfaces:

$$L_i = \frac{A(P, j, k)}{A(i, j, k)}$$

Point P defined inside the element with planar co-ordinates $[L_1, L_2, L_3]$ is transformed to the Cartesian system with co-ordinates $[x, y]$ according to:

$$\begin{aligned} x_P &= x_1.L_1 + x_2.L_2 + x_3.L_3 \\ y_P &= y_1.L_1 + y_2.L_2 + y_3.L_3 \\ I &= L_1 + L_2 + L_3 \end{aligned}$$

It follows that point 1 of the element has planar coordinates of $[1, 0, 0]$, point 2 $[0, 1, 0]$, and point 3 $[0, 0, 1]$.



Inverse relating to antecedent 3 equations represent equations for calculation of planar co-ordinates of the point P , defined in the Cartesian system:

$$L_i = (a_i + b_i.x + c_i.y) / (2.A), \text{ accordingly for } j, k.$$

Surface of the element A could be defined e.g. according to:

$$A = \frac{1}{2} \cdot \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\begin{aligned}
a_i &= x_j \cdot y_k - x_k \cdot y_j \\
b_i &= y_j - y_k \\
c_i &= x_k - x_j
\end{aligned}$$

we will obtain expressions for a_j, b_j, \dots, c_k by cyclic exchange of indexes.

We derive each function defined in the system of planar co-ordinates $f_{(L_1, L_2, L_3)}$ for needs of stiffness matrix, vector of volume forces and internal forces as composite function according to:

$$\begin{aligned}
\frac{\partial f_{(L_1, L_2, L_3)}}{\partial x} &= \frac{\partial f}{\partial L_1} \cdot \frac{\partial L_1}{\partial x} + \frac{\partial f}{\partial L_2} \cdot \frac{\partial L_2}{\partial x} + \frac{\partial f}{\partial L_3} \cdot \frac{\partial L_3}{\partial x} = \frac{1}{2 \cdot \Delta} \left(b_1 \cdot \frac{\partial f}{\partial L_1} + b_2 \cdot \frac{\partial f}{\partial L_2} + b_3 \cdot \frac{\partial f}{\partial L_3} \right) \\
\frac{\partial f_{(L_1, L_2, L_3)}}{\partial y} &= \frac{\partial f}{\partial L_1} \cdot \frac{\partial L_1}{\partial y} + \frac{\partial f}{\partial L_2} \cdot \frac{\partial L_2}{\partial y} + \frac{\partial f}{\partial L_3} \cdot \frac{\partial L_3}{\partial y} = \frac{1}{2 \cdot \Delta} \left(c_1 \cdot \frac{\partial f}{\partial L_1} + c_2 \cdot \frac{\partial f}{\partial L_2} + c_3 \cdot \frac{\partial f}{\partial L_3} \right)
\end{aligned}$$

Integration:

Calculation of integral is in this case simple, we advance according to Fellippe equation: Implicitly analytically.

$$\int_A L_1^p \cdot L_2^q \cdot L_3^r \cdot dA = \frac{p! \cdot q! \cdot r!}{(p + q + r + 2)!} \cdot 2 \cdot A$$

Thick plates

Element is based on assumption of Reissner-Mindlin's theory:

- Normal lines to the centreline plane remain straight also after deformation, however, they are not upright to the centreline plane of the plate, but there is neglected deplanation of cross section,
- Normal line of stress σ_z is comparing to stresses σ_x and σ_y negligible small, such as in the Kirchhoff theory. Of this condition their is also in this theory contradiction with $\varepsilon_z = 0$. This phenomenon originates by reduction of task dimension.

Vector of deformation $\varepsilon = [\varepsilon_x, \varepsilon_y, 0, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T$ is composed of deflexion deformation of surface $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ that are linearly changed by high of the cross section and constant slope by high of the cross section γ_{yz}, γ_{zx} .

$$\kappa = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_y}{\partial x} \\ -\frac{\partial \phi_x}{\partial y} \\ \frac{\partial \phi_y}{\partial y} - \frac{\partial \phi_x}{\partial x} \end{bmatrix} \quad \gamma = \begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x} + \phi_y \\ \frac{\partial w}{\partial y} - \phi_x \end{bmatrix}$$

While w, ϕ_x and ϕ_y are independent variable, contrary of the the Kirchhoff plate theory.

The element could bet physically isotropic, as well as orthotropic, shape of physical matrix D_B is:

$$D_B = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 \\ & & D_{33} & 0 & 0 \\ & & & D_{44} & D_{45} \\ \text{sym.} & & & & D_{55} \end{bmatrix}$$

In case of isotropy there is valid:

$$D_{13} = D_{23} = D_{45} = 0$$

$$D_{11} = D_{22} = \frac{E.h^3}{12.(1-\mu^2)}$$

$$D_{33} = \frac{1-\mu}{2} \cdot \frac{E.h^3}{12.(1-\mu^2)}$$

$$D_{12} = \frac{\mu.E.h^3}{12.(1-\mu^2)}$$

$$D_{44} = D_{55} = \frac{\pi^2}{12} \cdot \frac{E.h}{2.(1+\mu)}$$

Material orthotropy is represented by angle of rotation of angle β orthotropy to the planar coordinate system of the element.

LYNN - DHILLON

Element defined in the Cartesian co-ordinates with linear approximation functions with adding of quadratic beams for deflexion for improvement of convergence. Problem with shear locking at decreasing thickness is solved by introduction of numeric-stabilisation test: $(D_{44}, D_{55}) \leq 500 \cdot (D_{11}, D_{22}) / A$, where A is planar scope of the element.

Number of nodes:

3

Coordinates of nodes:

$\{0, 0\}, \{x_2, y_2\}, \{x_3, y_3\}$

Shape functions:

$$w = a_1 + a_2 \cdot x + a_3 \cdot y - \frac{1}{2} \cdot a_8 \cdot x^2 + \frac{1}{2} \cdot (a_5 - a_8) \cdot x \cdot y + \frac{1}{2} \cdot a_6 \cdot y^2$$

$$\omega_x = a_4 + a_5 \cdot x + a_6 \cdot y$$

$$\omega_y = a_7 + a_8 \cdot x + a_9 \cdot y$$

Integration:

Implicitly analytically

MITC4

Isoparametric linear element according to [Bathe], elimination of shear locking at decreasing thickness is done by mixed interpolation of deflexion, rotation and slope.

Qualities of the element:

- element matrix is obtained by full Gauss's numeric integration ,

- element has no zero mode of energy

Number of nodes:

4

Unit coordinates of nodes:

$\{-1, -1\}, \{1, -1\}, \{1, 1\}, \{-1, 1\}$

Shape functions:

$$N_1 = 1/4 \cdot (1 - \xi) \cdot (1 - \eta)$$

$$N_2 = 1/4 \cdot (1 + \xi) \cdot (1 - \eta)$$

$$N_3 = 1/4 \cdot (1 + \xi) \cdot (1 + \eta)$$

$$N_4 = 1/4 \cdot (1 - \xi) \cdot (1 + \eta)$$

Numeric integration:

Full (2 x 2)

Integration of points and weight multiplier:

$$\xi = [1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3} \quad -1/\sqrt{3}]$$

$$\eta = [1/\sqrt{3} \quad -1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3}]$$

$$w = [1 \quad 1 \quad 1 \quad 1]$$

Facet-shell structures

Facet-shell structure is an operating term for planar (2-dimensional) structures situated in 3-dimensional space and loaded so that there is not possible in them to separate the plate impact from the wall impact. There is used an element from planar tension. Bending element depends on selected plate theory. The final tension vector: $\sigma = [\sigma_x, \sigma_y, 0, \tau_{xy}, \tau_{yz}, \tau_{zx}]^T$, deformation vector: $\varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T$.

Internal forces are composed of a vector of σ membrane stresses and vector of m bending moments and shear forces.

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} = \tau_{yx} \end{bmatrix} \quad m = \begin{bmatrix} m_x \\ m_y \\ m_{xy} = m_{yx} \\ q_x \\ q_y \end{bmatrix}$$

The matrix of physical constants of the element D is composed of a bending part D_B , membrane part D_M . Material orthotropy is defined by the angle of the rotation of the angle β of orthotropy to planar co-ordinate system of the element, into which are because of so defined element transformed physical quantities.

In case of physical non-linearity, by origination of cracks along the height of a section will come to moving of the position of bearing of centreline plane in comparing to the original one, which divided the width of the element into 2 equal parts. This effect respects a sub matrix D_{BM} , it describes cohesion of wall and plate effects:

$$D = \begin{bmatrix} D_B & D_{BM} \\ sym. & D_M \end{bmatrix}$$

Large deformations

In solving of geometrically non-linear tasks by the Neton-Raphson method is used quadratic element of Green vector in deformation ε_{II} , which includes non-linear terms that are neglected in the theory of small deformations:

Subsoil

Every finite element can have along the entire surface continuous contact with effective model of a subsoil of the Kolar-Nemec type, which is defined by five constants in planar coordinates $[x^p, y^p, z^p]$:

$$C_{1x}, C_{1y}, C_{1z}, C_{2x}, C_{2y}.$$

Relevant forces for deformations are membrane (r_u, r_v) like and bending (r_w, t_x, t_y) like.

$$\begin{aligned} r_u^p &= C_{1x} \cdot u^p & t_x^p &= C_{2x} \cdot \frac{\partial w^p}{\partial x^p} \\ r_v^p &= C_{1y} \cdot v^p & t_y^p &= C_{2y} \cdot \frac{\partial w^p}{\partial y^p} \\ r_w^p &= C_{1z} \cdot w^p \end{aligned}$$

The constants C_{1x}, C_{1y} express the resistance against planar movements of centreline plane of the element (friction). The constants C_{1z}, C_{2x}, C_{2y} are coefficients of relations expressing resistance of surrounding against the movement and angular rotation. The constant C_{1z} responds to the Winkler model, C_{2x}, C_{2y} Pasternak model. In most cases is $C_{1x} = C_{1y}$ and $C_{2x} = C_{2y}$.

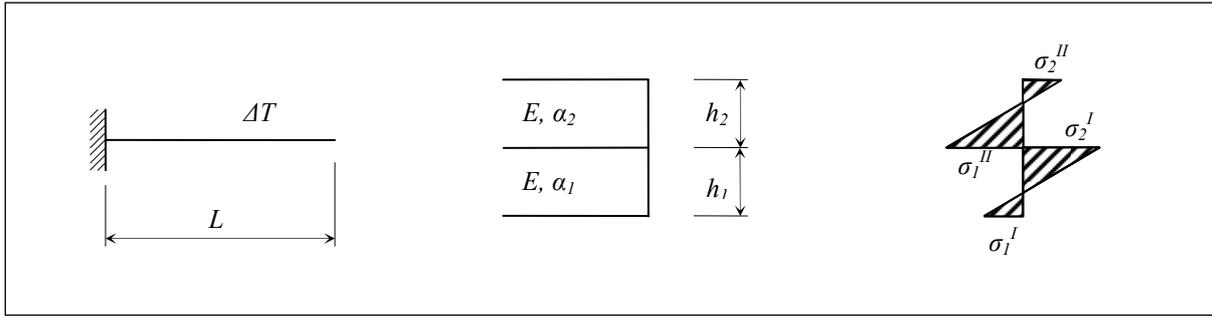
Sandwich elements

The element is along the height divided into a few (at least 2) isotropic, or orthotropic layers, ideally resistant linked together, so that there does not come to shearping. Every layer has its physical parameters, which are possible to solve so far only by physical linear elasticity. According to Newton-Raphson is possible to solve such elements by geometrical non-linearity. The number of layers is unlimited. The thickness of every layer is constant. Load of an element is analogous, as well as with facet-shell elements.

Alternate homogeneous cross-section with ideal static quantities, which are dealt with in calculation will be created from a composite cross-section. For specification we alternate resistance come out of equality of work of internal and external forces. Evaluation of internal forces comes off in optional point along the height of a cross-section, where on the basis of relevant equations of elasticity is finished calculation of final quantities.

Example:

Bimetallic strip - console, constant change of the temperature. $E=3e+7$ [Pa], $\mu=0$, $h_1=h_2=0.05$ [m], $\Delta T=100$ [K], $\alpha_1=1e-5$, $\alpha_2=2e-5$. The length of a beam $L=10$ [m], the width 1 [m]. (VM 35).



Picture. No. xx

Geometry, cross-section of an element and results – process of tensions along the height of a cross-section.

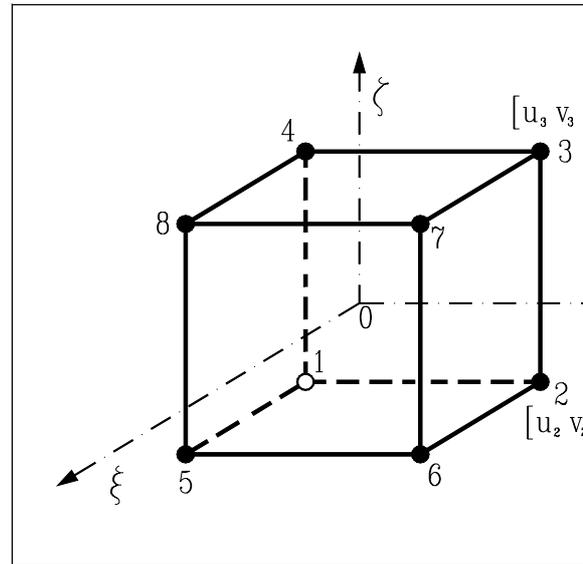
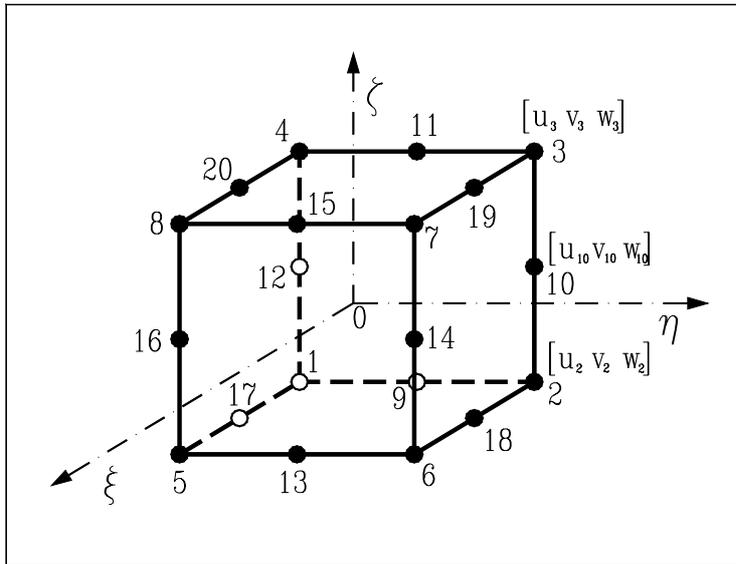
Deformations and tension	u_x [m]	u_z [m]	σ_1^I [Pa]	σ_2^I [Pa]	σ_1^{II} [Pa]	σ_2^{II} [Pa]
numerically	0.15	0.75	-7500	15000	-15000	7500
analytically	0.15	0.75	-7500	15000	-15000	7500

Chart No. xx
Results.

SPACE ELEMENTS

3-DIMENZIONAL ISOPARAMETRIC ELEMENTS WITH ROTATIONAL DEGREES OF FREEDOM (SERENDIPITY FAMILY)

Every vertex of an element has 6 degrees of freedom: $[u, v, w, \varphi_x, \varphi_y, \varphi_z]$. Elements do not have nodal -point in the centre of sides, edges can be straight only. Elements are defined by geometry of apexes and isotropic material. In case of 6-nodal -point element and 8-nodal -point is possible to use material orthotropy, that is used in contact elements.



Picture No.xx

20-nodal -point original 60 DOF brick without rotational degrees of freedom.

8- nodal -point new 48 DOF brick with rotational degrees of freedom.

Is used numeric gauss integration. Every element has relevant number of integral points, while because of the fastness of calculation is used reduced integration. This results in origination of zero energy states, which are treated by additional resistances. We distinguish 2 types: equal rotations and hourglass. For every element 's area is used analogical technique of additional resistance introduction into resistant matrix of an element so as for 2D membrane elements.

For all isoparametrical elements is used equal principle of transformation of nodal -point deformations in centres of sides into vertices so as it is in the case of 2-dimensional elements, but it is completed by the third dimension. Movements in centres of the sides

$$u_k = \frac{1}{2} \cdot (u_i + u_j) + \frac{y_j - y_i}{8} \cdot (\omega_{zj} - \omega_{zi}) + \frac{z_j - z_i}{8} \cdot (\omega_{yj} - \omega_{yi})$$

are eliminated by mediation:

$$w_k = \frac{1}{2} \cdot (w_i + w_j) + \frac{x_j - x_i}{8} \cdot (\omega_{yj} - \omega_{yi}) + \frac{y_j - y_i}{8} \cdot (\omega_{xj} - \omega_{xi})$$

$$v_k = \frac{1}{2} \cdot (v_i + v_j) + \frac{z_j - z_i}{8} \cdot (\omega_{xj} - \omega_{xi}) + \frac{x_j - x_i}{8} \cdot (\omega_{zj} - \omega_{zi})$$

Space elements with smaller number of vertices than 8 are solved in 2 ways:

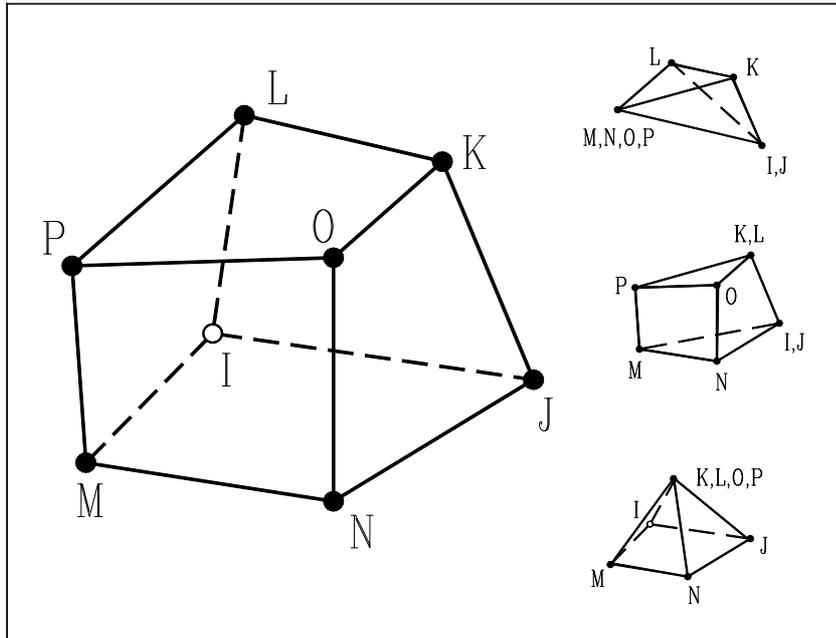
1/ by degeneration of 8-nodal -point element, picture No.xx,

2/ by defining of own shape functions and by position of integral points.

Both techniques are implemented, for a user is available only the technique No.2.

Hereinafter is in detail described only this technique.

8-nodal-point element - brick is derived from quadratic izoparametric element with 20 nodal- points. Such an element allows using curved edges, its disadvantage is large width of a half -belt and consequently large consumption of machine time. By above mentioned elimination of nodal-points in centres of sides of 20-nodal-point brick was achieved equations system with tighter half-bend, what lead to acceleration of calculation. From the original stiffness matrix of the 60x60 element will arise a matrix of the 48x48 size.



Picture No.xx
Scheme of brick degeneration.

Stiffness matrix

Hereinafter presented shape functions N create together with matrix of differential operators of δ 6 geometrical equations:

$$\delta^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$\varepsilon = \delta.N.u = B.u$, while:

$$\delta^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

Transformation of derivations of non-dimensional co-ordinates to Cartesian co-ordinates is:

Inverse relation of solution of equations:

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \frac{1}{J} \cdot \begin{bmatrix} \frac{\partial y}{\partial \eta} \cdot \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \eta} \cdot \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \xi} \cdot \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial z}{\partial \zeta} & \frac{\partial y}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \cdot \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \eta} \cdot \frac{\partial z}{\partial \zeta} & \frac{\partial x}{\partial \xi} \cdot \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \xi} \cdot \frac{\partial x}{\partial \zeta} & \frac{\partial z}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \eta} \cdot \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \zeta} & \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} \end{bmatrix} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix}$$

Where J is determinant of Jacob matrix of transformation:

$$J = J(\xi, \eta, \zeta) = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} \cdot \frac{\partial z}{\partial \zeta} - \frac{\partial x}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} \cdot \frac{\partial y}{\partial \zeta} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} \cdot \frac{\partial x}{\partial \zeta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} \cdot \frac{\partial z}{\partial \zeta} + \frac{\partial z}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \zeta} - \frac{\partial z}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} \cdot \frac{\partial x}{\partial \zeta}$$

Evaluation of Jacobean in any point:

$$J = \sum_{i=1}^M \sum_{j=1}^M \sum_{k=1}^M x_i \cdot y_j \cdot z_k \cdot \left(\frac{\partial N_i}{\partial \xi} \cdot \frac{\partial N_j}{\partial \eta} \cdot \frac{\partial N_k}{\partial \zeta} - \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} \cdot \frac{\partial N_j}{\partial \zeta} + \frac{\partial N_j}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} \cdot \frac{\partial N_i}{\partial \zeta} - \frac{\partial N_j}{\partial \xi} \cdot \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial N_k}{\partial \zeta} + \frac{\partial N_k}{\partial \xi} \cdot \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial N_j}{\partial \zeta} - \frac{\partial N_k}{\partial \xi} \cdot \frac{\partial N_j}{\partial \eta} \cdot \frac{\partial N_i}{\partial \zeta} \right)$$

Loading vector

$$\text{Vector of volume forces: } f_v = - \int_V [N] \cdot [b] \cdot dV$$

Vector of initial deformation of temperature has in case of 3D shape:

$$\begin{aligned} \{\varepsilon^{tep}\} &= \Delta T \cdot [\alpha_x \quad \alpha_y \quad \alpha_z \quad 0 \quad 0 \quad 0]^T \\ f_e &= - \int_V [B]^T \cdot [D] \cdot [\varepsilon^{tep}] \cdot dV \end{aligned}$$

Geometric matrix

Linear part of the deformation matrix B_L , is assemble on base of matrix of differential operators δ , introduced at previous text. Non-linear part B_{NL} , necessary for solving of stability (or for calculation of tangent matrix) is of shape:

$$\tilde{B}_{NL} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & \dots \\ \frac{\partial N_1}{\partial y} & 0 & 0 & \frac{\partial N_2}{\partial y} & \dots \\ \frac{\partial N_1}{\partial z} & 0 & 0 & \frac{\partial N_2}{\partial z} & \dots \end{bmatrix}$$

Cauchy's matrix of stresses:

$$\tilde{\tau} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

On base of presented assumptions we are able to assemble geometric matrix K_G , with which we could approach to solving of eigennumbers:

$$|K_L - \lambda.K_G|_{\det} = 0$$

Mass matrix

This matrix is not diagonal in most of cases. However, it is symmetric, positively defined. We convert it to diagonal one to mass vector by thereafter described algorithm.

We adjust mass matrix in general:

$$K_M = \rho \cdot \int_V [N]^T \cdot [N] dV$$

to shape of vector according to algorithm:

1/ we calculate

$$M_i = \sum_{j=1}^N K_{M(i,j)} \quad \text{for } i = 1, N$$

2/ set up

$$\begin{aligned} K_{M(i,j)} &= 0 & \text{for } i < j \\ K_{M(i,i)} &= M_i & \text{for } i = 1, N \end{aligned}$$

Stress

Results - stresses – we will obtain directly in integrating Gauss points. For each element we will reach field of stresses:

$\sigma_g = [\sigma_{x1}, \dots, \sigma_{xN}, \sigma_{y1}, \dots, \sigma_{yN}, \sigma_{z1}, \dots, \sigma_{zN}, \tau_{xy1}, \dots, \tau_{xyN}, \tau_{yz1}, \dots, \tau_{yzN}, \tau_{zx1}, \dots, \tau_{zxN}]$, N is number of integration points. Each element has specific number of integration points and their position in dimensionless co-ordination system ξ, η, ζ .

Taking into account evaluation of results in 3D, it is more convenient to provide user results directly in vertices σ_v , what we solve with their extrapolation to vertices of brick tranship of virtual hyperbolic function F that is different for each type of element. We will reach transformation matrix T_σ , that assure extrapolation of results to vertices, from solutions of equations of function F in corresponding integration points:

$$\sigma_v = T_\sigma \cdot \sigma_g.$$

For solving of stability we operate with resulting stress in brick that we obtain as an arithmetical mean of vertices.

Tetrahedron

Number of nodes:

10

Unit coordinates of nodes:

$\{0, 0, 0\}, \{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}$

Shape functions:

$$N_1 = (1 - \xi - \eta - \zeta) \cdot (1 - 2 \cdot \xi - 2 \cdot \eta - 2 \cdot \zeta)$$

$$N_2 = \xi \cdot (2 \cdot \xi - 1)$$

$$N_3 = \eta \cdot (2 \cdot \eta - 1)$$

$$N_4 = \zeta \cdot (2 \cdot \zeta - 1)$$

$$N_5 = 4 \cdot \xi \cdot (1 - \xi - \eta - \zeta)$$

$$N_6 = 4 \cdot \eta \cdot (1 - \xi - \eta - \zeta)$$

$$N_7 = 4 \cdot \zeta \cdot (1 - \xi - \eta - \zeta)$$

$$N_8 = 4 \cdot \xi \cdot \eta$$

$$N_9 = 4 \cdot \eta \cdot \zeta$$

$$N_{10} = 4 \cdot \zeta \cdot \xi$$

Numeric integration:

Selective, reduced (6-point), 5 for $\varepsilon_x, \varepsilon_y, \varepsilon_z$, 1 for $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
integration points and weight multiplier:

$$\begin{matrix} \xi = [1/6 & 1/2 & 1/6 & 1/6 & 1/4 & 1/4] \\ \eta = [1/6 & 1/6 & 1/2 & 1/6 & 1/4 & 1/4] \\ \zeta = [1/6 & 1/6 & 1/6 & 1/2 & 1/4 & 1/4] \\ w = [9/20 & 9/20 & 9/20 & 9/20 & -4/5 & 1] \end{matrix}$$

Evaluation:

Function of surface for extrapolation of qualities of integration points to vertices:

$$F = a_1 + a_2 \cdot \xi + a_3 \cdot \eta + a_4 \cdot \zeta$$

Pyramid element

Number of nodes

13

Unit coordinates of nodes:

$\{-1, -1, 0\}, \{1, -1, 0\}, \{1, 1, 0\}, \{-1, 1, 0\}, \{0, 0, 1\}$

Shape of the function

$$\begin{matrix} N_1 = 1/4 \cdot (-\xi - \eta - 1) \cdot ((1 - \zeta) \cdot (1 - \eta) - \zeta + (\xi \cdot \eta \cdot \zeta) / (1 - \zeta)) \\ N_2 = 1/4 \cdot (\xi - \eta - 1) \cdot ((1 + \zeta) \cdot (1 - \eta) - \zeta - (\xi \cdot \eta \cdot \zeta) / (1 - \zeta)) \\ N_3 = 1/4 \cdot (\xi + \eta - 1) \cdot ((1 + \zeta) \cdot (1 + \eta) - \zeta + (\xi \cdot \eta \cdot \zeta) / (1 - \zeta)) \\ N_4 = 1/4 \cdot (-\xi + \eta - 1) \cdot ((1 - \zeta) \cdot (1 + \eta) - \zeta - (\xi \cdot \eta \cdot \zeta) / (1 - \zeta)) \\ N_5 = \zeta \cdot (2 \cdot \zeta - 1) \\ N_6 = (1 + \xi - \zeta) \cdot (1 - \xi - \zeta) \cdot (1 - \eta - \zeta) / (2 \cdot (1 - \zeta)) \\ N_7 = (1 + \eta - \zeta) \cdot (1 - \eta - \zeta) \cdot (1 + \xi - \zeta) / (2 \cdot (1 - \zeta)) \\ N_8 = (1 + \xi - \zeta) \cdot (1 - \xi - \zeta) \cdot (1 + \eta - \zeta) / (2 \cdot (1 - \zeta)) \\ N_9 = (1 + \eta - \zeta) \cdot (1 - \eta - \zeta) \cdot (1 - \xi - \zeta) / (2 \cdot (1 - \zeta)) \\ N_{10} = \zeta \cdot (1 - \xi - \zeta) \cdot (1 - \eta - \zeta) / (1 - \zeta) \\ N_{11} = \zeta \cdot (1 + \xi - \zeta) \cdot (1 - \eta - \zeta) / (1 - \zeta) \\ N_{12} = \zeta \cdot (1 + \xi - \zeta) \cdot (1 + \eta - \zeta) / (1 - \zeta) \\ N_{13} = \zeta \cdot (1 - \xi - \zeta) \cdot (1 + \eta - \zeta) / (1 - \zeta) \end{matrix}$$

Numeric integration

reduced 8 - point (2 x 2 x 2)

integration of points and weight multiplier:

$$\begin{matrix} \psi_1 = 0.50661630334978769, \psi_2 = 0.26318405556971380 \\ \psi_3 = 0.12251482265544100, \psi_4 = 0.54415184401122496 \\ \psi_5 = 0.23254745125350801, \psi_6 = 0.10078588207982500 \end{matrix}$$

$$\begin{matrix} \xi = [-\psi_1 & \psi_1 & \psi_1 & -\psi_1 & -\psi_2 & \psi_2 & \psi_2 & -\psi_2] \\ \eta = [-\psi_1 & -\psi_1 & \psi_1 & \psi_1 & -\psi_2 & -\psi_2 & \psi_2 & \psi_2] \\ \zeta = [\psi_3 & \psi_3 & \psi_3 & \psi_3 & \psi_4 & \psi_4 & \psi_4 & \psi_4] \\ w = [\psi_5 & \psi_5 & \psi_5 & \psi_5 & \psi_6 & \psi_6 & \psi_6 & \psi_6] \end{matrix}$$

Evaluation:

Function of surface for extrapolation of qualities of integration points to vertices:

$$F = a_1 + a_2 \cdot \xi + a_3 \cdot \eta + a_4 \cdot \zeta + a_5 \cdot \xi \cdot \eta + a_6 \cdot \eta \cdot \zeta + a_7 \cdot \zeta \cdot \xi + a_8 \cdot \xi \cdot \eta \cdot \zeta$$

Triangular prism (wedge)

Number of nodes

15

Unit coordinates of nodes: $\{0, 0, -1\}, \{1, 0, -1\}, \{0, 1, -1\}, \{0, 0, 1\}, \{1, 0, 1\}, \{0, 1, 1\}$ **Shape of the function**

$$N_1 = \frac{1}{2} \cdot (1 - \zeta) \cdot (1 - \zeta - \eta) \cdot (-2 \cdot \zeta - 2 \cdot \eta - \zeta)$$

$$N_2 = \frac{1}{2} \cdot \zeta \cdot (1 - \zeta) \cdot (2 \cdot \zeta - \zeta - 2)$$

$$N_3 = \frac{1}{2} \cdot \eta \cdot (1 - \zeta) \cdot (2 \cdot \eta - \zeta - 2)$$

$$N_4 = \frac{1}{2} \cdot (1 + \zeta) \cdot (1 - \zeta - \eta) \cdot (-2 \cdot \zeta - 2 \cdot \eta + \zeta)$$

$$N_5 = \frac{1}{2} \cdot \zeta \cdot (1 + \zeta) \cdot (2 \cdot \zeta + \zeta - 2)$$

$$N_6 = \frac{1}{2} \cdot \eta \cdot (1 + \zeta) \cdot (2 \cdot \eta + \zeta - 2)$$

$$N_7 = 2 \cdot \zeta \cdot (1 - \zeta) \cdot (1 - \zeta - \eta)$$

$$N_8 = 2 \cdot \zeta \cdot \eta \cdot (1 - \zeta)$$

$$N_9 = 2 \cdot \eta \cdot (1 - \zeta) \cdot (1 - \zeta - \eta)$$

$$N_{10} = 2 \cdot \zeta \cdot (1 + \zeta) \cdot (1 - \zeta - \eta)$$

$$N_{11} = 2 \cdot \zeta \cdot \eta \cdot (1 + \zeta)$$

$$N_{12} = 2 \cdot \eta \cdot (1 + \zeta) \cdot (1 - \zeta - \eta)$$

$$N_{13} = (1 - \zeta - \eta) \cdot (1 - \zeta^2)$$

$$N_{14} = \zeta \cdot (1 - \zeta^2)$$

$$N_{15} = \eta \cdot (1 - \zeta^2)$$

Numeric integration

reduced 6 - point (3 x 2)

integration points and weight multiplier:

$$\begin{array}{l} \xi = [1/6 \quad 2/3 \quad 1/6 \quad 1/6 \quad 2/3 \quad 1/6] \\ \eta = [1/6 \quad 1/6 \quad 2/3 \quad 1/6 \quad 1/6 \quad 2/3] \\ \zeta = [1/\sqrt{3} \quad -1/\sqrt{3} \quad -1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3}] \\ w = [1/3 \quad 1/3 \quad 1/3 \quad 1/3 \quad 1/3 \quad 1/3] \end{array}$$

Evaluation:

Function of surface for extrapolation of qualities of integration points to vertices:

Function of extrapolation is obtained by combination of triangle and line element:

 $(b_1 + b_2 \cdot \zeta + b_3 \cdot \eta), (b_4 + b_5 \cdot \zeta)$, their multiplying and adjusting to shape:

$$F = a_1 + a_2 \cdot \zeta + a_3 \cdot \eta + a_4 \cdot \zeta + a_5 \cdot \zeta \cdot \zeta + a_6 \cdot \eta \cdot \zeta$$

8-nodal brick**Number of nodes**

20

Unit coordinates of nodes: $\{-1, -1, -1\}, \{1, -1, -1\}, \{1, 1, -1\}, \{-1, 1, -1\}, \{-1, -1, 1\}, \{1, -1, 1\}, \{1, 1, 1\}, \{-1, 1, 1\}$ **Shape of the function**

$$N_1 = \frac{1}{8} \cdot (1 - \xi) \cdot (1 - \eta) \cdot (1 - \zeta) \cdot (-\xi - \eta - \zeta - 2)$$

$$N_2 = \frac{1}{8} \cdot (1 + \xi) \cdot (1 - \eta) \cdot (1 - \zeta) \cdot (\xi - \eta - \zeta - 2)$$

$$N_3 = \frac{1}{8} \cdot (1 + \xi) \cdot (1 + \eta) \cdot (1 - \zeta) \cdot (\xi + \eta - \zeta - 2)$$

$$N_4 = \frac{1}{8} \cdot (1 - \xi) \cdot (1 + \eta) \cdot (1 - \zeta) \cdot (-\xi + \eta - \zeta - 2)$$

$$N_5 = \frac{1}{8} \cdot (1 - \xi) \cdot (1 - \eta) \cdot (1 + \zeta) \cdot (-\xi - \eta + \zeta - 2)$$

$$N_6 = \frac{1}{8} \cdot (1 + \xi) \cdot (1 - \eta) \cdot (1 + \zeta) \cdot (\xi - \eta + \zeta - 2)$$

$$N_7 = \frac{1}{8} \cdot (1 + \xi) \cdot (1 + \eta) \cdot (1 + \zeta) \cdot (\xi + \eta + \zeta - 2)$$

$$N_8 = 1/8 \cdot (1 - \xi) \cdot (1 + \eta) \cdot (1 + \zeta) \cdot (-\xi + \eta + \zeta - 2)$$

$$N_9 = 1/4 \cdot (1 - \xi^2) \cdot (1 - \eta) \cdot (1 - \zeta)$$

$$N_{10} = 1/4 \cdot (1 + \xi) \cdot (1 - \eta^2) \cdot (1 - \zeta)$$

$$N_{11} = 1/4 \cdot (1 - \xi^2) \cdot (1 + \eta) \cdot (1 - \zeta)$$

$$N_{12} = 1/4 \cdot (1 - \xi) \cdot (1 - \eta^2) \cdot (1 - \zeta)$$

$$N_{13} = 1/4 \cdot (1 - \xi^2) \cdot (1 - \eta) \cdot (1 + \zeta)$$

$$N_{14} = 1/4 \cdot (1 + \xi) \cdot (1 - \eta^2) \cdot (1 + \zeta)$$

$$N_{15} = 1/4 \cdot (1 - \xi^2) \cdot (1 + \eta) \cdot (1 + \zeta)$$

$$N_{16} = 1/4 \cdot (1 - \xi) \cdot (1 - \eta^2) \cdot (1 + \zeta)$$

$$N_{17} = 1/4 \cdot (1 - \xi) \cdot (1 - \eta) \cdot (1 - \zeta^2)$$

$$N_{18} = 1/4 \cdot (1 + \xi) \cdot (1 - \eta) \cdot (1 - \zeta^2)$$

$$N_{19} = 1/4 \cdot (1 + \xi) \cdot (1 + \eta) \cdot (1 - \zeta^2)$$

$$N_{20} = 1/4 \cdot (1 - \xi) \cdot (1 + \eta) \cdot (1 - \zeta^2)$$

Numeric integration

reduced 8 - point reduced (2 x 2 x 2)

integration points and weight multiplier:

$$\begin{matrix} \xi = [& -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}] \\ \eta = [& -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}] \\ \zeta = [& -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}] \\ w = [& 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1] \end{matrix}$$

Evaluation:

Function of surface for extrapolation of qualities from integration points to vertices:

$$F = a_1 + a_2 \cdot \xi + a_3 \cdot \eta + a_4 \cdot \zeta + a_5 \cdot \xi \cdot \eta + a_6 \cdot \eta \cdot \zeta + a_7 \cdot \xi \cdot \zeta + a_8 \cdot \xi \cdot \eta \cdot \zeta$$

EXAMPLES

For comparison we have selected following examples.

Example No. 1

Cantilever fig.No. 3, loaded by axial force. Cross section $b \times h = 0.3 \times 0.75$ m, length $L = 6$ m, $E = 20$ GPa, $\mu = 0.2$, density $\rho = 2300$ kg/m³, $N = 1000$ kN. We have monitored displacement on end (in the point 1), stress on beginning of cantilever in center of gravity of the brick (in the point 2). Stability was solved from axial force, dynamics from own weight.



Figure No.3

Cantilever loaded on free end by axial force.

dividing	u_1 [mm]	σ_2 [MPa]	λ	ω [rad.s ⁻¹]
1x1x6	1.458	4.44	2.44	0.310
2x2x15	1.332	4.44	2.37	0.311
2x4x30	1.331	4.44	2.301	0.311
analytically	1.333	4.44	2.313	0.314

Table No.1
Results.

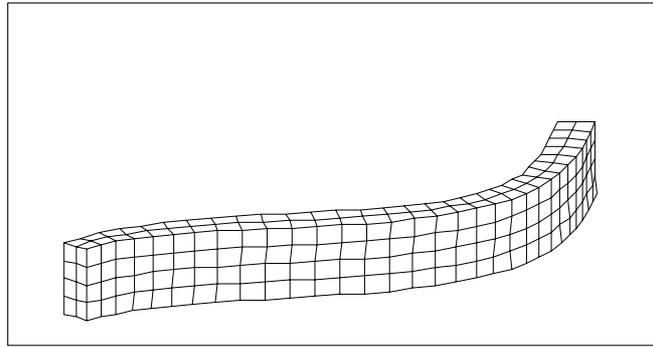
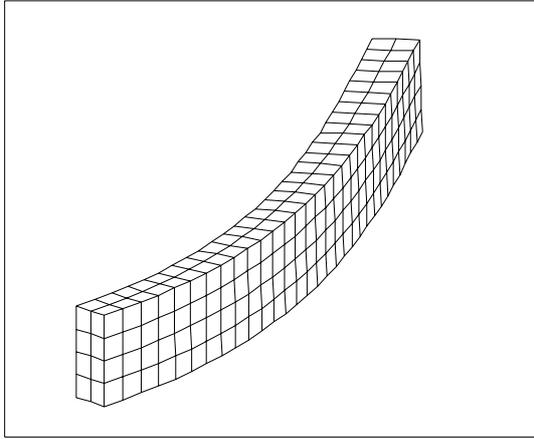


Figure No.4, 5
Stability – 1st and 4th own shape of buckling.

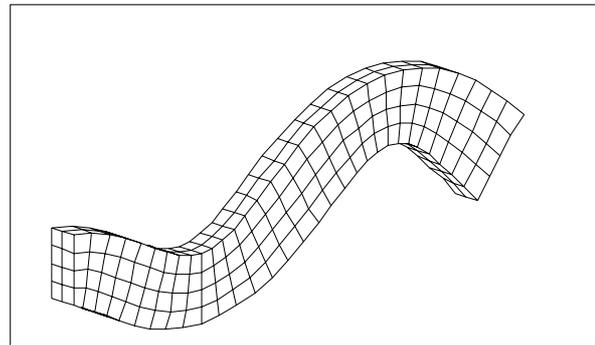
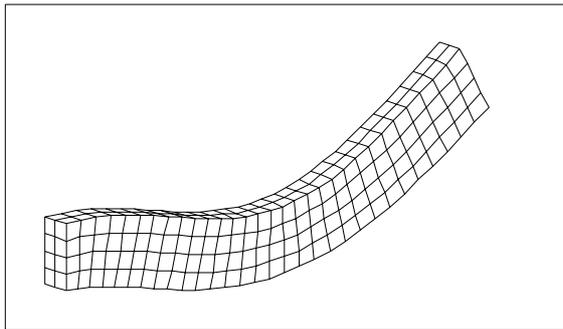


Figure No.6, 7
Dynamics - 3rd and 5th own shape of oscillation.

Example No. 2

Cantilever fig. No. xx, loaded by bending moment, assignment as in the example No.1, $M = 1000$ kNm. We have monitored deflexions on end (in the point 1), stress in bracket placing at upper surface of the construction (in the point 2).

Figure No.8
Bracket loaded on the free end by bending moment.

dividing	w_1 [m]	σ_2 [MPa]
1x1x6	0.08157	36.8592
2x2x12	0.08390	35.4808
2x4x18	0.08475	35.4047
analytically	0.08533	35.5555

Table No.2
Results.

Example No. 3

Plate Fig. No. 9, $a \times b = 3.5 \times 5$ m, thickness $h = 0.25$ m, $E = 20$ GPa, $\mu = 0.15$, uniform load $q = 250$ kNm⁻². We have monitored deflexions in the centre of plate (in the point 1), deflexions in the centre of free edge (in the point 2), moment in the centre plate (point 1) in direction of shorter side.

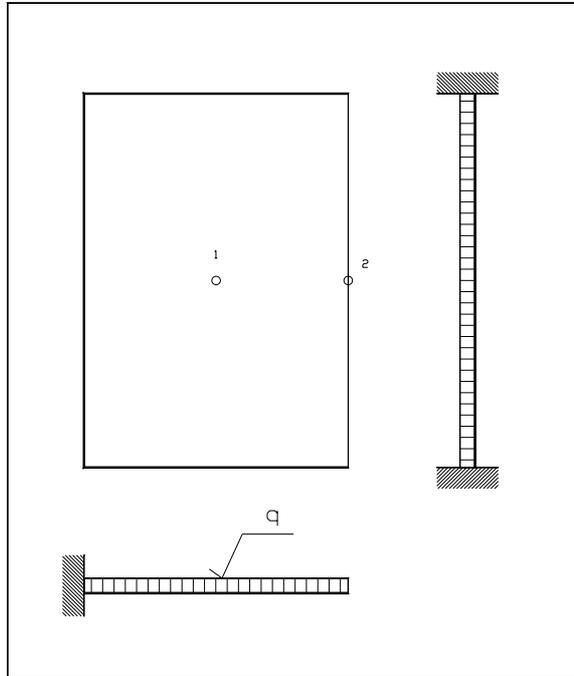


Figure No.9
Uniform loaded plate.

dividing	w_1 [m]	w_2 [m]	M_{x1} [kN.m]
14x20x2	0.00793	0.01483	57.5
analytically	0.00809	0.01454	58.2

Table No.3
Results.

Example No. 4

Plate Fig. No. 10, $a \times b = 6 \times 4$ m, thickness $h = 0.2$ m, $E = 20$ GPa, $\mu = 0.15$, $\rho = 2300$ kgm⁻³, supported along entire circumference fixing. We have solved own frequency and own shapes.

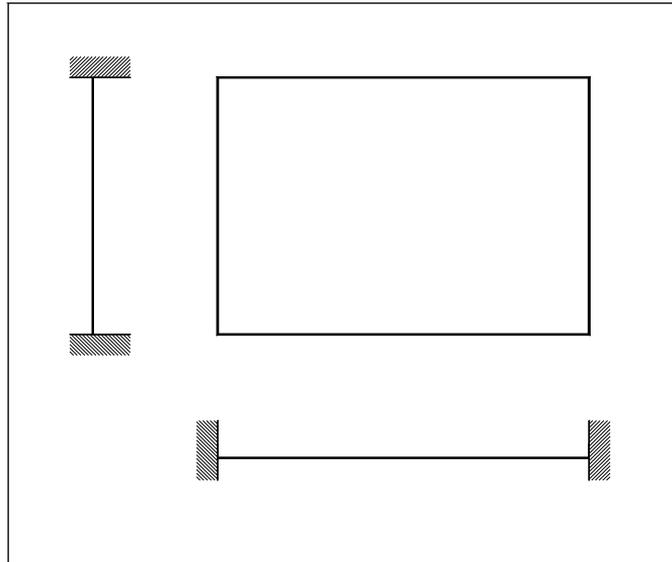


Figure No.10
Plate fixed along circumference.

dividing	f_1 [Hz]	f_2 [Hz]	f_3 [Hz]
6x4x1	54.10	90.68	162.7
12x8x1	46.98	72.53	116.9
60x40x2	45.61	70.05	110.1
analytically	45.974	71.145	112.58

Table No.4
Results.

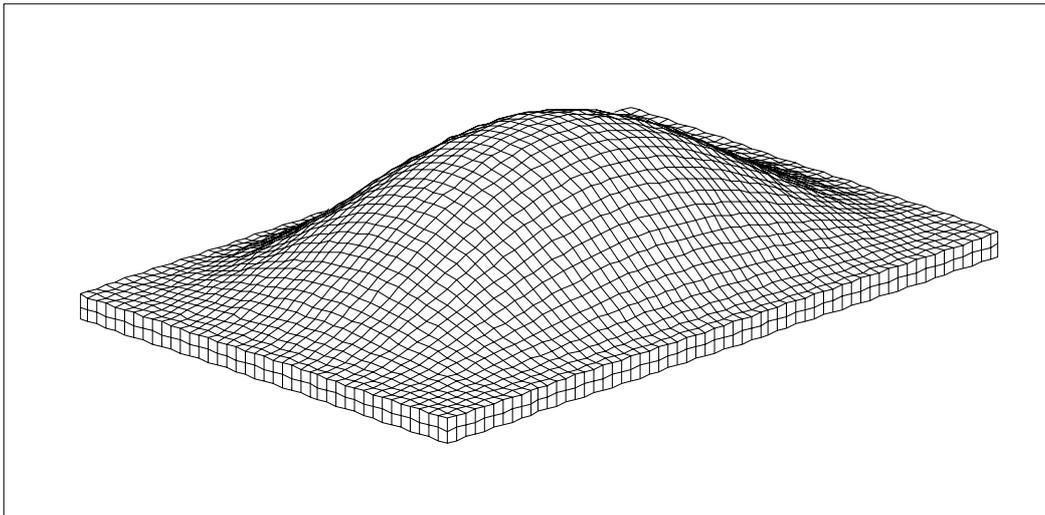


Figure No.11
Dynamics - 1st own shape of oscillation.

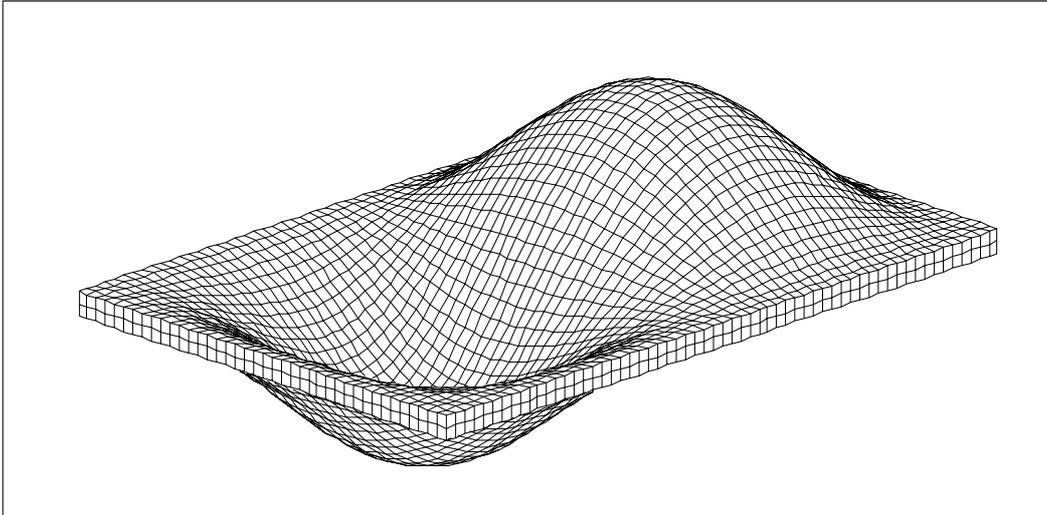


Figure No.12
Dynamics - 2nd own shape of oscillation.

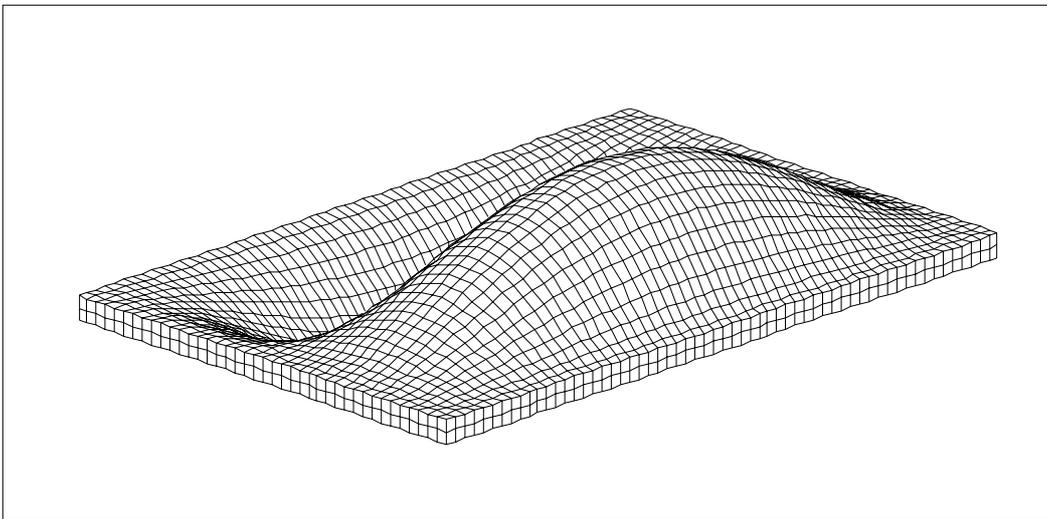


Figure No.13
Dynamics - 3rd own shape of oscillation.

Example No. č. 5

Plate Fig. No.14, $A \times B = 9 \times 3$ [m], thickness $h = 0.1$ [m], $E = 32.5$ [GPa], $\mu = 0.0$, supported along entire circumference articulated. Uniform membrane loading $q_n = -3.33$ [MN.m²], resultant $N = -1$ [MN]. We have solved critical loading and own shapes. Wall buckling in square valves - this shape represents minimal resistance against buckling.

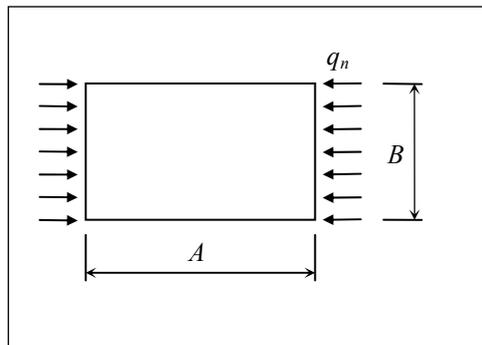


Figure No.xx
Plate along circumference fulcrumed, axially loaded.

dividing	N_{cr} [MN]
18x6x2	42.87
45x15x2	35.50
90x30x2	34.71
analytically	33.01

Table No.x
Results.

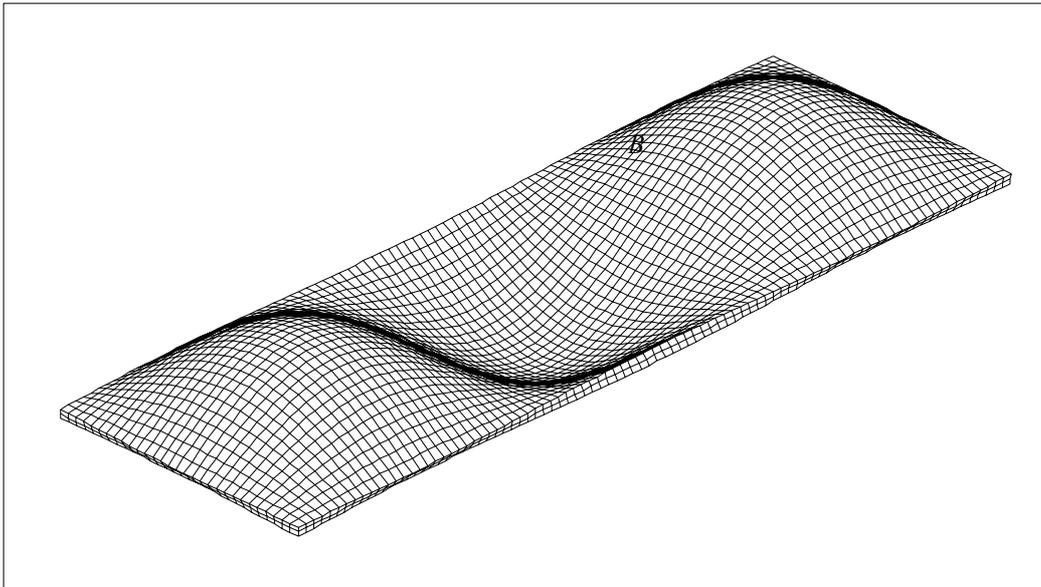


Figure No.xx
Stability - 1. own shape of buckling.

Example No. 6

Plate of previous example. Loading uniform shear $q_t = 100$ [kN.m⁻²] po circumference. We have solved critical loading and own shapes.

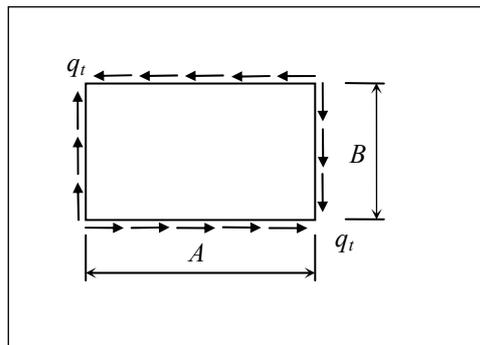


Figure No.xx
Plate by circumference fulcrumed, loaded by shear flow.

dividing	Q_{cr} [MN.m ⁻¹]
18x6x2	24.26

45x15x2	17.76
analytically	17.46

Table No.x
Results.

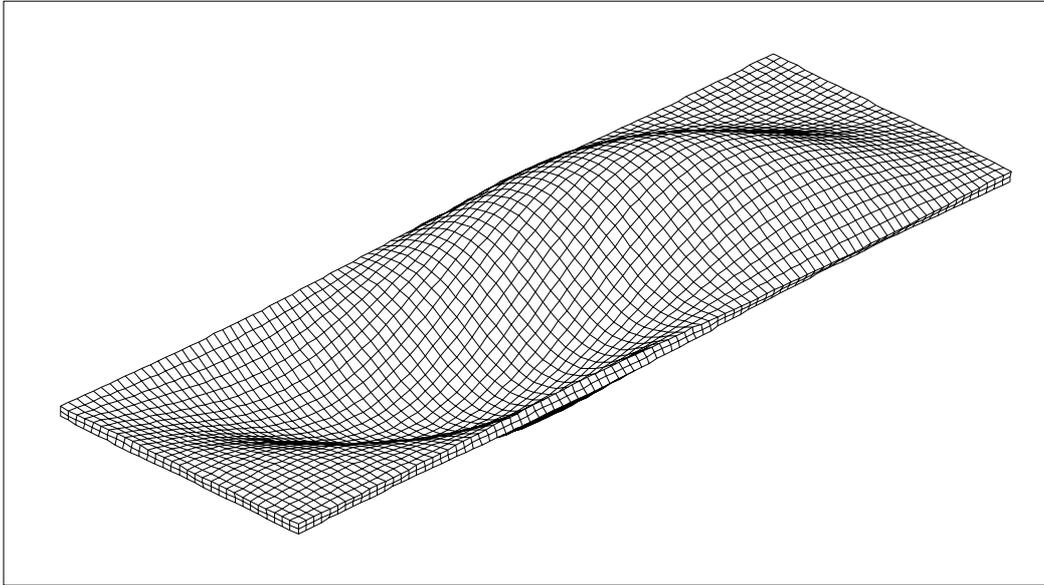


Figure No.xx
Stability – 1st own shape of buckling.

Example No. 7

Element modelled by bricks, assignment of example c.xx.

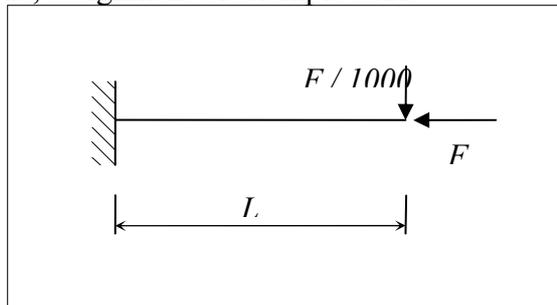
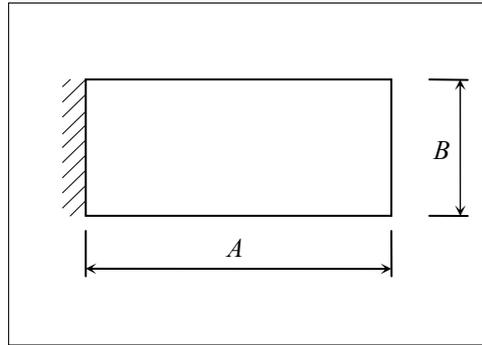


Figure No.xx
Beam modelled of 3D elements

dividing	u_z [mm]	ϕ_v [mrad]
1.00 x 1.00 x 1.00 [m]	7.692	1.173
0.50 x 0.50 x 0.50 [m]	7.745	1.180
Mathiasson	7.700	1.190

Table No.x
Results.



Example No. 8

Calculation of frequency of plate. Example was taken over of ANSYS Verification Manual, No. 66 - "Vibration of and flat plate". Used are imperial units. $A \times B = 16 \times 4$ [in], thickness $h = 1$ [in], $E = 30.10^6$ [psi], $\mu = 0.0$, specific mass $\rho = 0.000728$ [lb.sec².in⁻⁴]. Supports - fixing on the shorter side.

Figure c.xx
Scheme of plate.

Dividing				Timoshenko
4x2x1	8x2x1	16x4x1	32x8x2	128.09
125.57	127.16	127.56	127.76	

Table No.x
Results.

Example No. 9

Calculation of angular frequency ω . Example compares analytic solution of the column with continuously spreader mass with numeric one. Cross section of column $b \times h = 0.1 \times 0.2$ [m], $L = 6$ [m], $E = 27$ [GPa], $\mu = 0.0$, specific mass $\rho = 50$ [kg.m⁻³], what responds to 1 [kg.m⁻¹]. Solution of frequency equations ($\cos(\lambda).cosh(\lambda)+1=0$) was considered for first two shapes 1.875104 and 4.694091. Whereas calculation takes into account spatial acting, gained were $2 \times 2 = 4$ solutions.

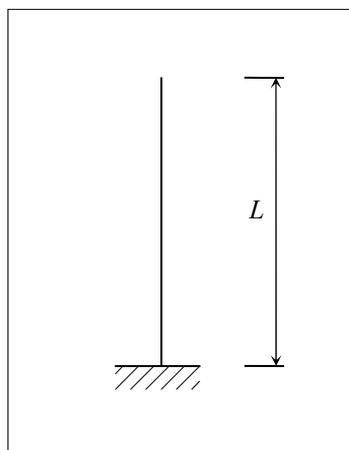


Figure c.xx
Scheme of column

Dividing	angular		frequency	ω [s ⁻¹]
6x1x1	2.0663	4.1205	12.9069	25.6399
12x2x2	2.0832	4.1448	12.9638	25.7887
24x2x2	2.0832	4.1448	12.9783	25.8178
48x4x2	2.0832	4.1448	12.9822	25.8379
analytically	2.0718	4.1436	12.9839	25.9676

Table No.x
Results.

Example No. 10

Calculation of angular frequency ω . Example is of literature Marton P., Travnicek F.: Dynamics of constitutional structures - Examples, STU Bratislava. For geometry of cross section was only moment of inertia $I_y = 1.6$ [m⁴]. $L = 1$ [m], $E = 210$ [GPa], $F = 2.2$ [kN]. Example does not take into account own mass of bracket. Mass of node was uniformly distributed by cross section in given point of acting.

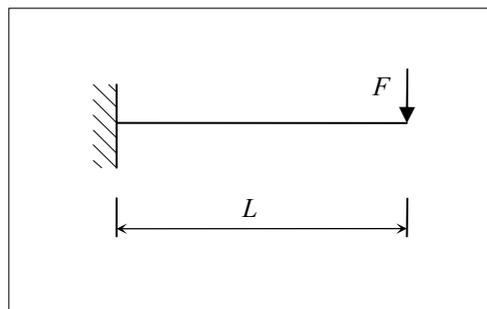


Figure c.xx
Scheme of the bracket

	Dividing	analytically
10x1x1	20x2x2	67.043
67.098	67.023	

Table No.x
Results - angular frequency ω [s⁻¹].

Example No.11

Rotation of the element as rigid unit. Brick supported by hinge in 1 node, loaded with force in opposite node, solved by geometric nonlinearity according to Newton-Raphson.

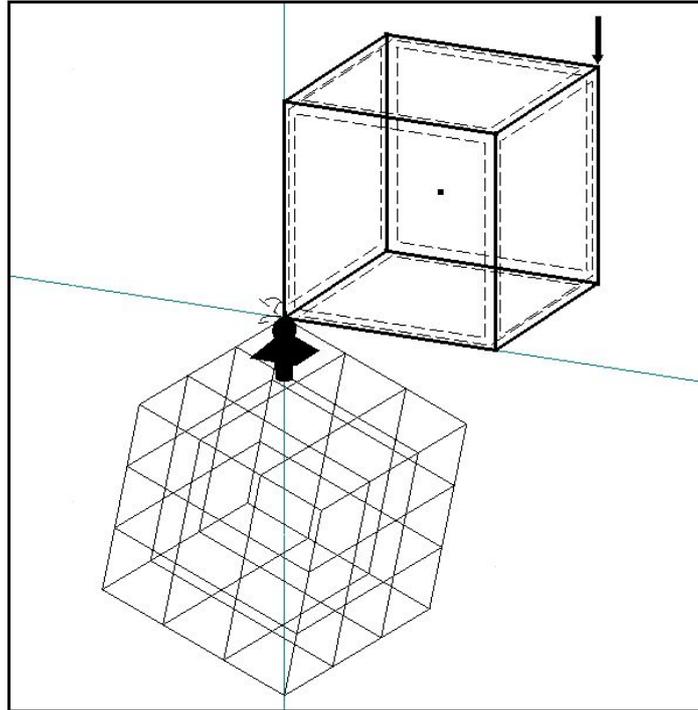


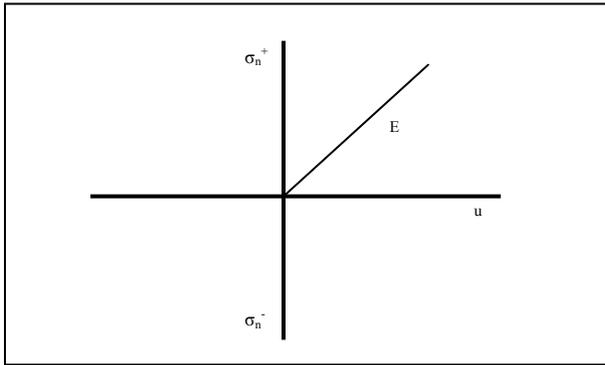
Fig.c.xx
Geometry and deformed shape.

CONTACTS

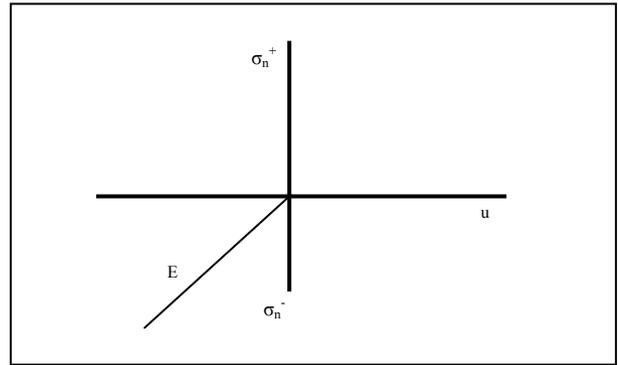
Basic assumption is, that during increases of load will not come to change of contact space of 2 elements, which have a contact among themselves. From that follows necessity of knowledge of a contact region before calculation. By this assumption is limited number of tasks, which are possible to solve. In the next stage of development can come to further development in this issue on the basis of user response.

Process of defining the contact is necessary except of geometry of contact areas, to choose adequate physical law of contact. This one is defined especially for normal treatment on contact plane and tangentially in its plane (friction). Normal influence does not cause bigger problems in calculation. On the contrary, by introduction of friction conditions in plane of the contact and mainly, which is dependent on normal force to the surface (Coloumb) mostly in the course of calculation causes problems with convergence, whole calculation can be markedly slowed down..

Normal contacts

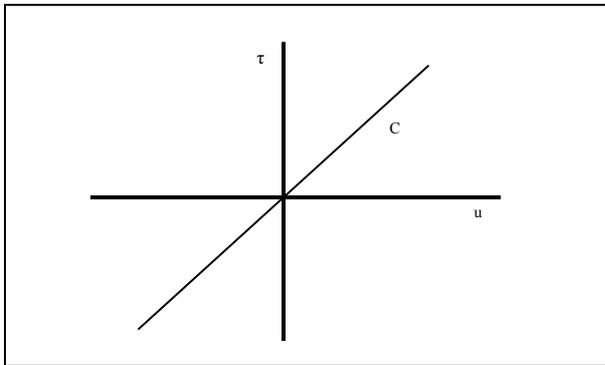


Picture No.xx
Exclusion of pressure

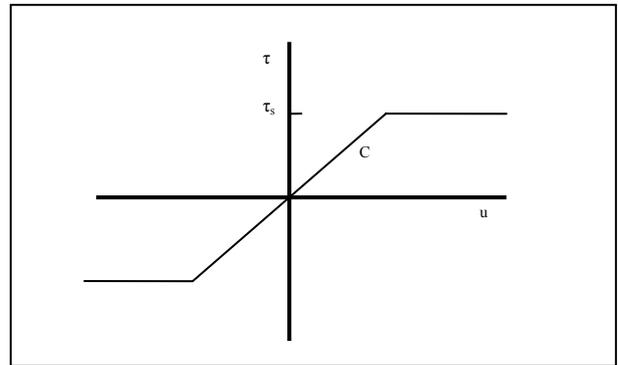


Picture No.xx
Exclusion of tension

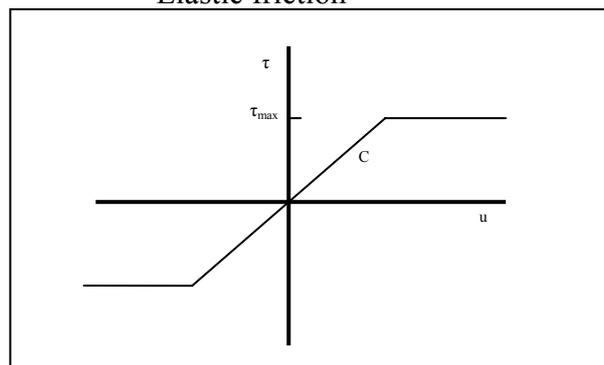
Tangential contacts



Picture No.xx
Elastic friction



Picture No.xx
Elastic Coulomb's friction



Picture No.xx
Elastic friction with τ_{\max} limitation

Key:

E - normal stiffness

C - stiffness in friction

σ_n - normal stress on plane of contact

$\tau_s = \mu \cdot |\sigma_n|$ - max. allowed tension in friction for Coulomb's friction

μ - coefficient of friction

τ_{max} - max. allowed tension in friction

POSTCRITICAL IMPACT

The method is defined for these types of constructions, in which will come to snap-through when is reached load in limit points and to following overleap on new stabile shape. In this way behave strut-framed beam, flat arcs and shells. Load tracks can compose of bifurcation points (bifurcation of balance). On the contrary, in consequence of used procedure of solving equals is important, that the load track would not contain reversions (snap-back).

Load tracks, which have only 1 limit point – standard case of stability - loading can reach its maximum (minimum) right in this point, collapse of a structure will occur, it has no sense to solve by this procedure.

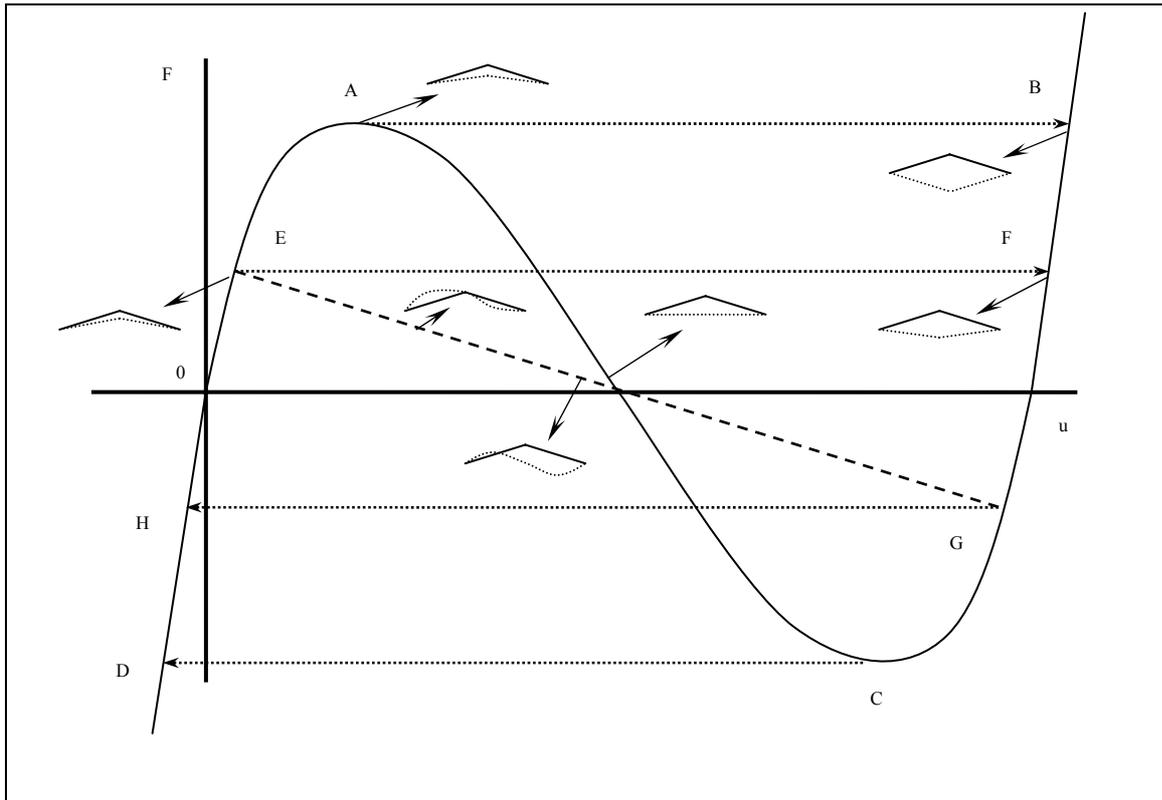
Description of strut-framed beam behaviour on the picture No.xx:

At first we consider a case of solid beams, where not bifurcation is points on the load track.

In gradual increasing of load from the initial point 0 we will reach the limit point A. In next slight enlargement of the load will come to sudden break of strut-framed beam into point B.

In this moment the grouped membrane potential will change into kinetic energy given by the capacity of the ABC space and this will cause dynamic change of the shape. During researching of a scientific notation of such an action we will find out, that at least 1 minor of the system is negative. Likewise, during reverse loading technique will come to movement from the point B into limit point C and to an overleap on the point D.

In case that axial force in a beam will exceed critical force of the beam, in load level in bifurcate point E will happen a leap on the point F and eventually further continuation along a curve.



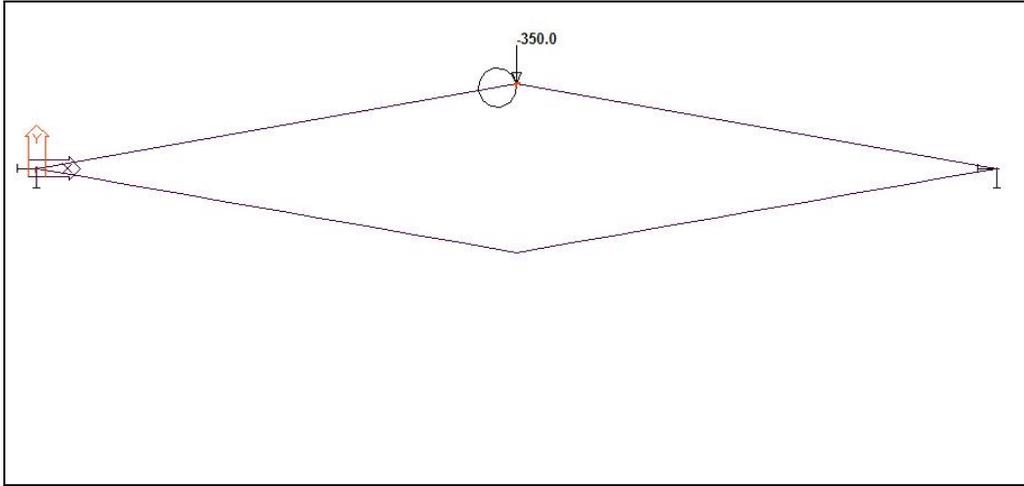
Picture No.xx
Loading track of strut-framed beam, effect of snap-through.

Description of behaviour of an arc on the picture No.xx:

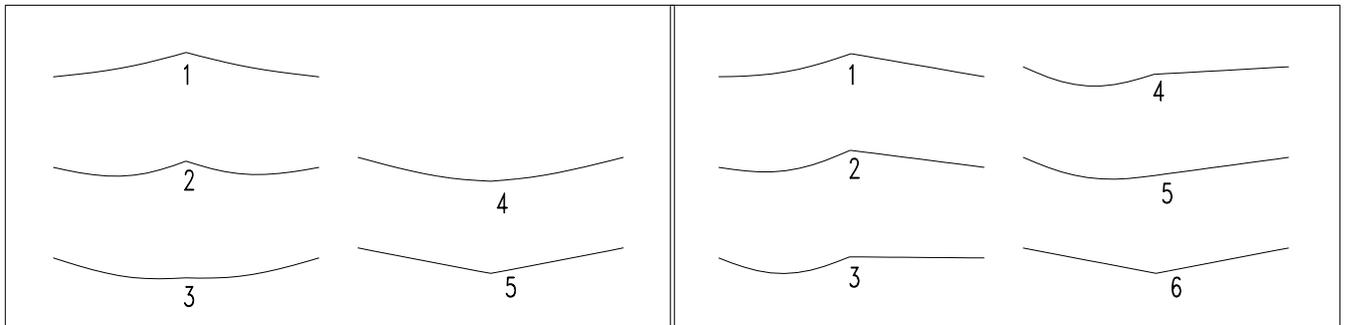
Loading track is analogical with strut-framed beam, but it contains a bifurcate point, in contrary to the strut-framed beam, which can, but need not contain this point.

During gradual increasing of loading from the initial point 0 we will reach the bifurcate point, while the level of loading is chosen so that we would get to close surrounding of the bifurcate point. It is appropriate to secure sufficient large number of load increases by so expressively geometrical non-linear task, so that we could get as precise idea of the structure response on loading as we can. While we are near the BB, by adding of small load the construction will lose shape, will curve, gain antimetric shape and without other adding of loading it will gain stabile shape (all minor systems are positive), while there comes to great deformations within 1 increase of loading.

But while increases of loading are like that, the system will pass without problems of the BB (probably with small number of increases of loading), we will get up to the limit point with solving , where the system is already singular. This state should not occur, it leads to symmetric deformations during overlap to a stabile branch. This case has little chance in practise.

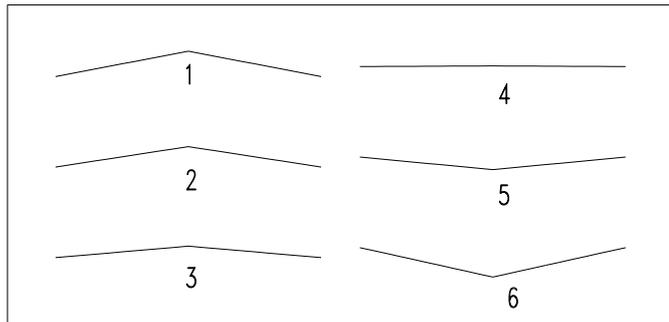


Picture No.xx
 Von-Mises girder modeled from 1D elements

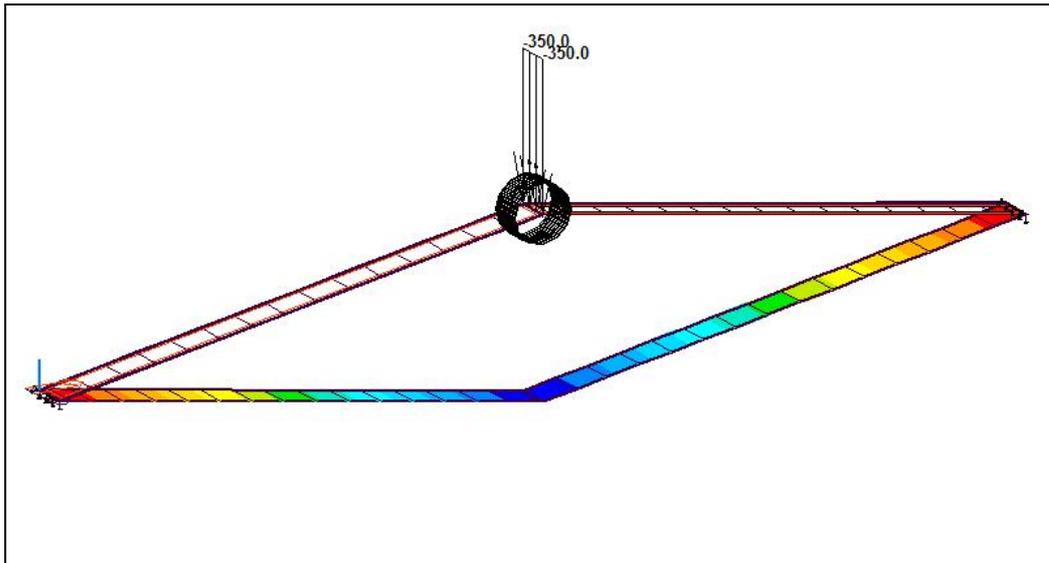


Picture No.xx
 Stiffness of the left beam = stiffness of the right beam, case No.1.

Picture No.xx
 Stiffness of the left beam = 4% of the right beam's stiffness, case No.2

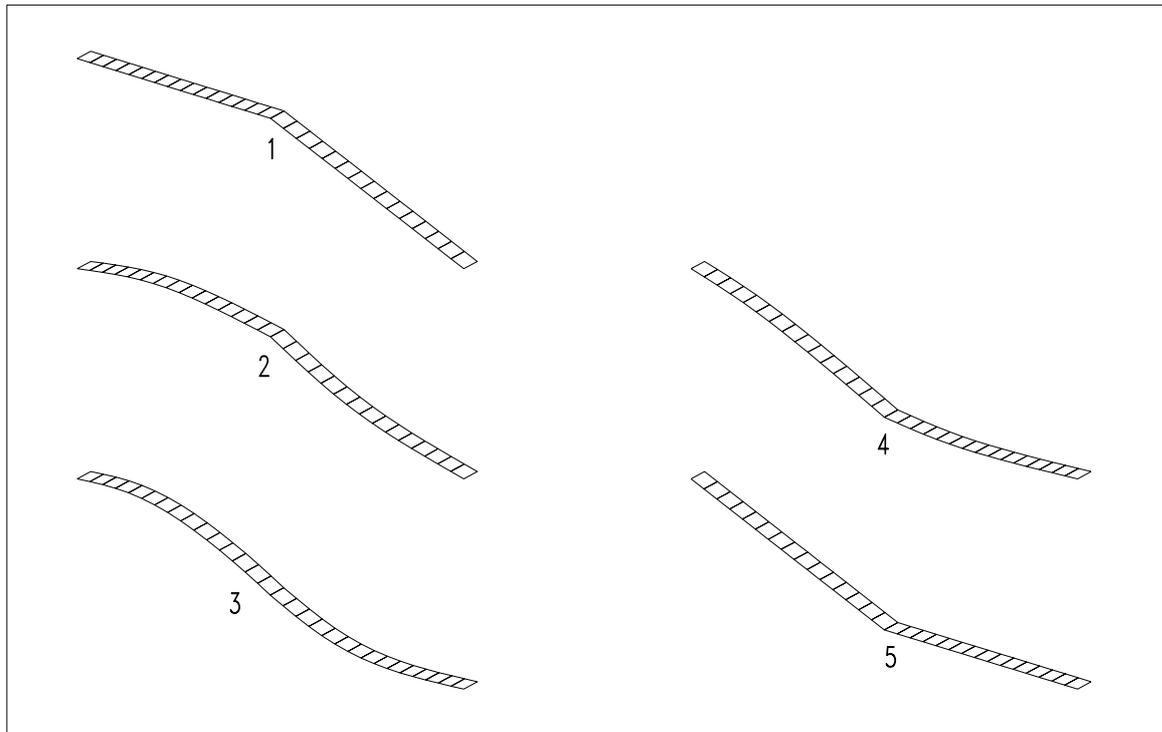


Picture No.xx
 Large stiffness of both beams, case No.3



Picture No.xx

Scheme of a strut-framed beam, modeled from 2D elements.

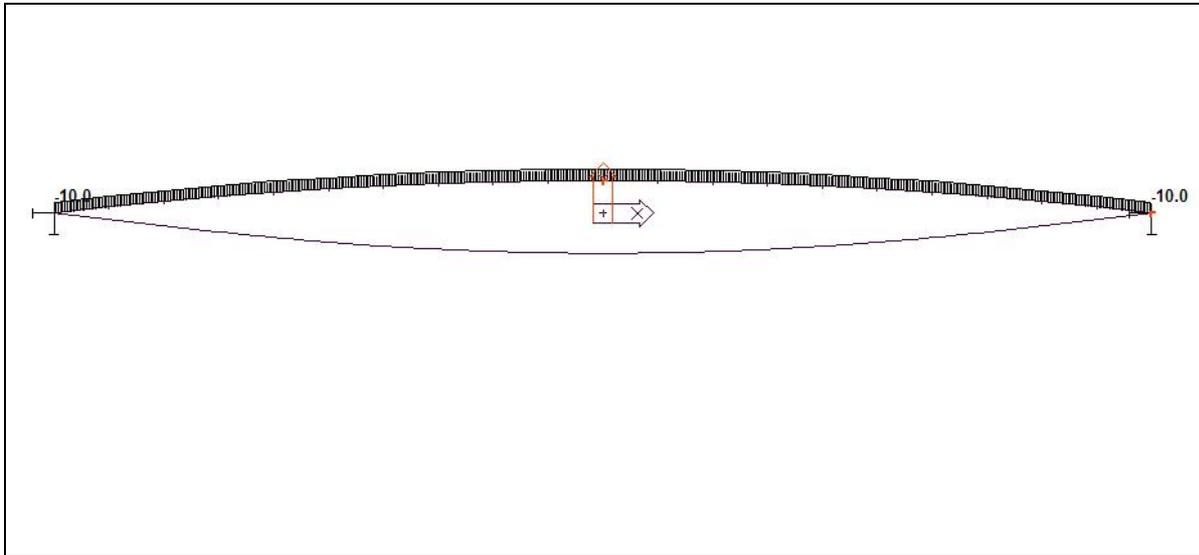


Picture No.xx

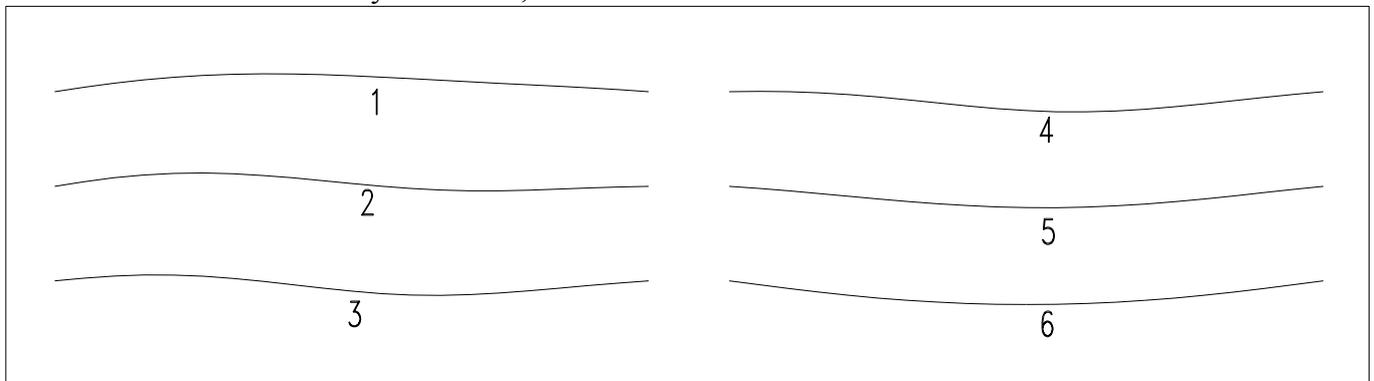
Von-Mises' s girder, at first both arms sheered during increasing of axial forces, only afterwards comes to change of general shape into final version. It is the case No.1.

Example No.2

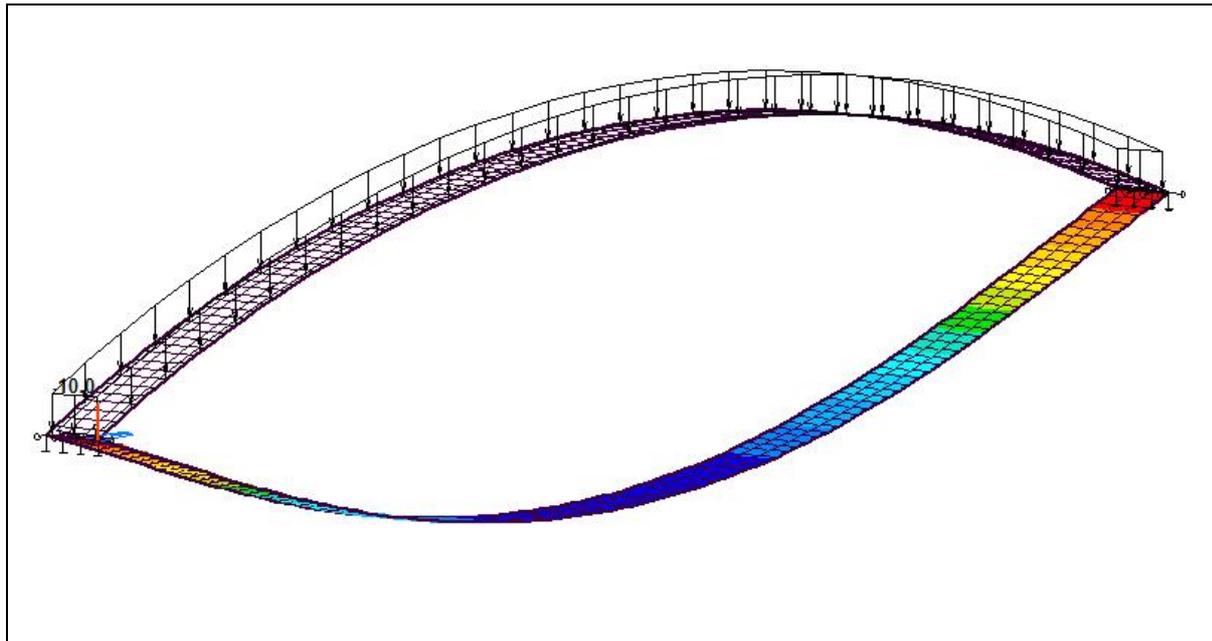
Flat arc hinged, loaded equally by force.



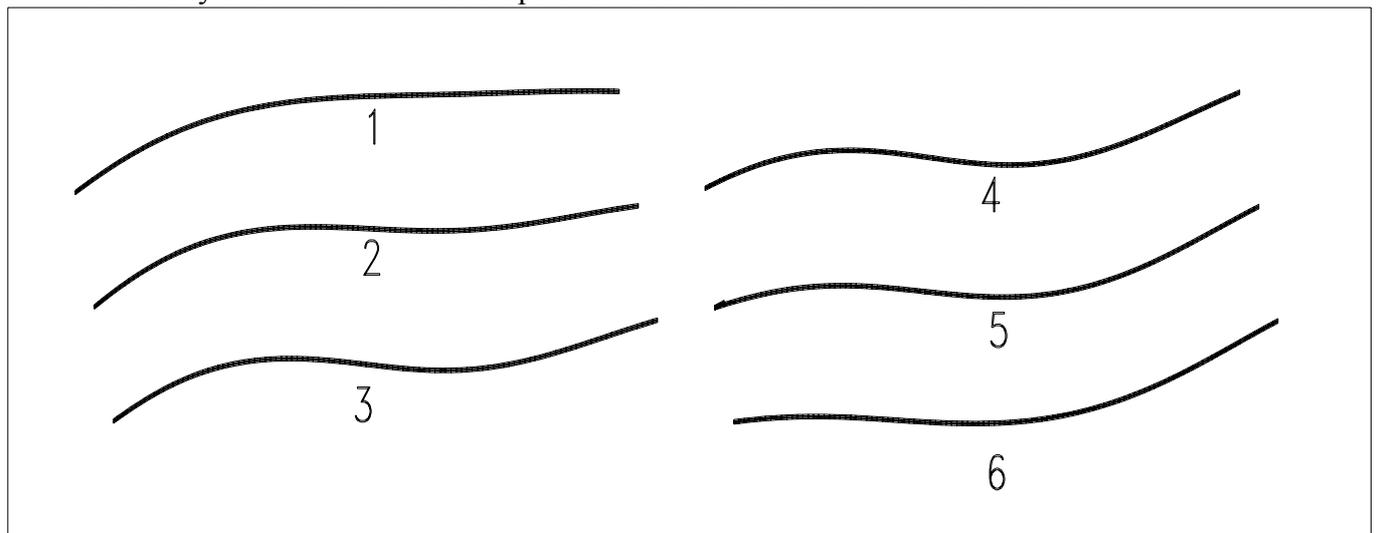
Picture No.xx
Geometry of flat arc, modelled from 1D elements.



Picture No.xx
Stages of transition from the original through unstable into resulting- stable shape, while gradual addition of load.



Picture No.xx
 Geometry and final deformed shape of a flat arc modelled from 2D elements.



Picture No.xx

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